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ON THE CONSTRUCTION OF GENERALIZED JACOBIANS

by LEI FU

ABSTRACT. We give a modern exposition of the construction of generalized jacobians using Weil's method.

0. INTRODUCTION

Generalized jacobians of algebraic curves are treated in detail in [S]. In this book Serre uses the terminology "generic points" that is developed in Weil's *Foundations of Algebraic Geometry*. Nowadays one uses the terminology in Grothendieck's *Éléments de Géométrie Algébrique*, and it is hard for students studying algebraic geometry to get used to Weil's terminology. At least my personal experience tells me so. So in this paper we use Weil's method and Grothendieck's language to construct generalized jacobians.

In § 1 we state a theorem of Grothendieck that is used throughout this paper. In § 2 we list some basic properties of relative effective Cartier divisors. We construct a birational group in § 3 and show how to get an algebraic group from a birational group in § 4. In § 5 we prove some fundamental properties of generalized jacobians. The main results are Theorems 1 and 2. In § 6 we prove that the generalized jacobian of a curve is the Picard scheme of the curve. The Appendix contains the proof of a technical proposition.

While preparing this note, I was helped by [A], [BLR], [Mi] and [S].

1. A THEOREM OF GROTHENDIECK

The following theorem is a special case of Grothendieck's theorems, and the proof can be found in [Mu] §5, [H] §3.12, or [EGA] III, §7.7.5, 7.9.4.

THEOREM 1.1. *Let $q\colon V \to T$ be a proper flat morphism of noetherian schemes and let $\mathcal{L}$ be an invertible sheaf on V. For each $t \in T$ denote the fiber $V \otimes_T \operatorname{spec}(k(t))$ of q at t by V_t, where $k(t)$ is the residue field of T at t. Denote the inverse image of $\mathcal{L}$ on V_t by $\mathcal{L}_t$.*

(a) *The function $t \mapsto \chi(\mathcal{L}_t) = \sum_i (-1)^i \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ is locally constant on T.*

(b) *For each i, the function $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ on T is upper semicontinuous.*

(c) *If T is reduced and connected and if $t \mapsto \dim_{k(t)} H^i(V_t, \mathcal{L}_t)$ is a constant function on T, then $R^i q_* \mathcal{L}$ is a locally free sheaf on T and the map $R^i q_* \mathcal{L} \otimes_{\mathcal{O}_T} k(t) \to H^i(V_t, \mathcal{L}_t)$ is an isomorphism.*

(d) *If $H^1(V_t, \mathcal{L}_t) = 0$ for all $t \in T$, then $R^1 q_* \mathcal{L} = 0$ and $q_* \mathcal{L}$ is a locally free sheaf. Moreover the formation of $q_* \mathcal{L}$ commutes with any base change.*

2. RELATIVE EFFECTIVE CARTIER DIVISORS

Let $q\colon X \to T$ be a morphism of noetherian schemes. A *relative effective Cartier divisor* on X/T is an effective Cartier divisor on X that is flat over T when regarded as a closed subscheme of X. When $T = \operatorname{spec}(R)$ is affine, a closed subscheme D of X is a relative effective Cartier divisor if and only if there exists an open affine covering $U_i = \operatorname{spec}(R_i)$ of X and $g_i \in R_i$ such that

(a) $D \cap U_i = \operatorname{spec}\big(R_i/(g_i)\big)$;

(b) g_i is not a zero divisor;

(c) $R_i/(g_i)$ is flat over R.

REMARK 2.1. Let D be an effective Cartier divisor on X/T, let $\mathcal{I}(D)$ be the sheaf of ideals defining D, and let $\mathcal{L}(D)$ be the invertible sheaf corresponding to D. We have $\mathcal{L}(D) = \mathcal{I}(D)^{-1}$. The inclusion $\mathcal{I}(D) \subset \mathcal{O}_X$ induces $\mathcal{O}_X \subset \mathcal{I}(D)^{-1} = \mathcal{L}(D)$, hence a section s_D of $\mathcal{L}(D)$.

The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf on X and s is a global section of $\mathcal{L}$ such that the map $s\colon \mathcal{O}_X \to \mathcal{L}$ induced by the section s is injective and $\mathcal{L}/s\mathcal{O}_X$ is $\mathcal{O}_T$-flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) *If D_1 and D_2 are relative effective Cartier divisors on X/T, then so is $D_1 + D_2$.*

(b) *Let D_1 and D_2 be two relative effective Cartier divisors on X/T and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be their ideal sheaves. If $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, then $D_1 - D_2$ is also a relative effective Cartier divisor on X/T.*

(c) *Let $T' \to T$ be a base extension and let $X' = X \times_T T'$. If D is a relative effective Cartier divisor on X/T, then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T'.*

LEMMA 2.3. *Assume $q\colon X \to T$ is flat. Let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_X$ and let D be the closed subscheme of X defined by $\mathcal{I}$. If for every point $x \in D$, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X,x}$ is generated by one element g_x whose image in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$ is not a zero divisor, then D is a relative effective Cartier divisor.*

Proof. It suffices to show that g_x is not a zero divisor in $\mathcal{O}_{X,x}$ and that $\mathcal{O}_{X,x}/(g_x)$ is flat over $\mathcal{O}_{T,q(x)}$. This follows from [EGA] §0.10.2.4 by taking $A = \mathcal{O}_{T,q(x)}$, $B = \mathcal{O}_{X,x}$, $M = N = \mathcal{O}_{X,x}$, and $u\colon M \to N$ to be the homomorphism $g_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ defined by the multiplication by g_x.

3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k. A *modulus* $\mathfrak{m}$ supported on a finite subset S of X is a divisor of the form $\mathfrak{m} = \sum_{P \in S} n_P P$ with each $n_P > 0$. For any rational function f on X, we write $f \equiv 0 \mod \mathfrak{m}$ if $v_P(f) \geq n_P$ for every $P \in S$, where v_P is the valuation defined by P. Two divisors D_1 and D_2 on X prime to S are called $\mathfrak{m}$-*equivalent* if there exists a rational function f satisfying $f - 1 \equiv 0 \mod \mathfrak{m}$ such that $D_1 - D_2 = (f)$. If this holds, we write $D_1 \sim_{\mathfrak{m}} D_2$. Define a ringed space $(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}})$ as follows: The underlying set of $X_{\mathfrak{m}}$ is $(X-S) \cup \{Q\}$. Define

$$\mathcal{O}_{X_{\mathfrak{m}},Q} = k + \{f \mid f \equiv 0 \mod \mathfrak{m}\}$$

and for every $x \in X - S$, define $\mathcal{O}_{X_{\mathfrak{m}},x} = \mathcal{O}_{X,x}$. One can show that when $\deg(\mathfrak{m}) \geq 2$, the ringed space $X_{\mathfrak{m}}$ is a singular curve with a unique singular point Q and its normalization is X. (It is easy to see that when $\deg(\mathfrak{m}) < 2$, the ringed space $X_{\mathfrak{m}}$ is identified with X itself.) For a divisor D of X prime to S, we put

$$L_{\mathfrak{m}}(D) = H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}), \quad I_{\mathfrak{m}}(D) = H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}),$$

where $\mathcal{L}_{\mathfrak{m}}$ is the invertible sheaf on $X_{\mathfrak{m}}$ corresponding to D. Denote the dimensions of $L_{\mathfrak{m}}(D)$ and $I_{\mathfrak{m}}(D)$ by $l_{\mathfrak{m}}(D)$ and $i_{\mathfrak{m}}(D)$, respectively. The Riemann-Roch theorem states that

$$l_{\mathfrak{m}}(D) - i_{\mathfrak{m}}(D) = \deg(D) + 1 - \pi.$$

In this formula, π is the sum $\pi = g + \delta$, where g is the genus of X and $\delta = \deg(\mathfrak{m}) - 1$. All these results are proved in [S], Chapter IV.

For convenience, a closed point on a scheme is just called a point.

Let T be a connected k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k). \end{array}$$

Since $X_{\mathfrak{m}}$ is proper and flat over $\operatorname{spec}(k)$, the morphism q is also proper and flat. Let D be a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$ supported on $(X_{\mathfrak{m}} - Q) \times T$ and let $\mathcal{L}$ be the invertible sheaf corresponding to D. Applying Theorem 1.1 (a) to the morphism q and the invertible sheaf $\mathcal{L}$, we conclude that $t \mapsto \chi(\mathcal{L}_t)$ is a constant function on T. By the Riemann-Roch theorem, we have $\chi(\mathcal{L}_t) = \deg D_t + 1 - \pi$. So $\deg(D_t)$ is also a constant. This constant is called the *degree* of D. Denote by $\operatorname{Div}^{(n)}(T)$ the set of all relative effective Cartier divisors of degree n on $(X_{\mathfrak{m}} \times T)/T$ supported on $(X_{\mathfrak{m}} - Q) \times T$.

Let $(X-S)^{(n)}$ be the n-th symmetric power of $X - S$, i.e., the quotient of $(X-S)^n$ by the action of the n-th symmetric group $\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $(X-S)^n$ by permuting the factors. In the Appendix we show that there exists a relative effective Cartier divisor $\mathcal{D} \in \operatorname{Div}^{(n)}\big((X-S)^{(n)}\big)$, called the *universal relative effective Cartier divisor*, whose restriction to the fiber of the projection $X_{\mathfrak{m}} \times (X-S)^{(n)} \to (X-S)^{(n)}$ at $P_1 + \cdots + P_n \in (X-S)^{(n)}$ is the divisor $P_1 + \cdots + P_n$ of $X_{\mathfrak{m}}$. Moreover, we have

PROPOSITION 3.1. *The functor* $T \mapsto \mathrm{Div}^{(n)}(T)$ *from the category of k-schemes to the category of sets is represented by the symmetric power* $(X-S)^{(n)}$. *More precisely, for any relative effective Cartier divisor* D *of degree* n *on* $(X_{\mathfrak{m}} \times T)/T$ *supported on* $(X_{\mathfrak{m}} - Q) \times T$, *there exists a unique morphism* $f: T \to (X-S)^{(n)}$ *such that the pull-back of* $\mathcal{D}$ *by* $\mathrm{id} \times f$ *is* D.

The proof of this proposition is given in the Appendix. The morphism $T \to (X-S)^{(n)}$ can be described as follows: For every $t \in T$, identifying the fiber of $q: X_{\mathfrak{m}} \times T \to T$ at t with $X_{\mathfrak{m}}$, we may regard the restriction D_t of D to the fiber at t as an effective divisor of degree n on $X_{\mathfrak{m}}$ supported on $X_{\mathfrak{m}} - Q$. But this kind of divisor can be thought of as a point in $(X-S)^{(n)}$. The morphism $T \to (X-S)^{(n)}$ is just $t \mapsto D_t$.

LEMMA 3.2. *Let* D *be a divisor of* X *prime to* S *such that* $i_{\mathfrak{m}}(D) \geq 1$. *Then there exists an open subset* U *of* $X-S$ *such that for every* $P \in U$, *we have* $i_{\mathfrak{m}}(D+P) = i_{\mathfrak{m}}(D) - 1$.

Proof. If $P \notin \mathrm{Supp}(D) \cup S$, then the dual vector space $I_{\mathfrak{m}}(D+P)^*$ of $I_{\mathfrak{m}}(D+P)$ is identified with the subspace of $I_{\mathfrak{m}}(D)^*$ formed by differential forms $\omega \in I_{\mathfrak{m}}(D)^*$ vanishing at P. Let $\{\omega_1, \ldots, \omega_{i_{\mathfrak{m}}(D)}\}$ be a basis of $I_{\mathfrak{m}}(D)^*$. We can then take U to be the complement of

$$\mathrm{Supp}(D) \cup S \cup \{P \mid \omega_i(P) = 0 \text{ for } i = 1, \ldots, i_{\mathfrak{m}}(D)\}.$$

LEMMA 3.3. *Let* D_0 *be a divisor of* X *prime to* S *of degree* 0. *Then the set*

$$V_{D_0} = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) = 1 \text{ and } l(D+D_0-\mathfrak{m}) = 0\}$$

is non-empty and open in $(X-S)^{(\pi)}$.

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ (X-S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

Applying Theorem 1.1 (b) to q and the invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} + p^*(D_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor, we conclude that the set

$$V_1 = \{t \in (X-S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \leq 1\}$$

is open, that is,

$$V_1 = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) \leq 1\}$$

is open. By the Riemann-Roch theorem we have, for any $D \in (X-S)^{(\pi)}$,

$$l_{\mathfrak{m}}(D+D_0) \geq \deg(D+D_0)+1-\pi = 1.$$

So we must have

$$V_1 = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) = 1\}.$$

If $l_{\mathfrak{m}}(D_0) \neq 0$, then there exists a rational function f on X such that $(f)+D_0$ is an effective divisor on X prime to S. This effective divisor must be 0 since it is of degree 0. Hence $l_{\mathfrak{m}}(D_0) = l_{\mathfrak{m}}((f)+D_0) = l_{\mathfrak{m}}(0) = 1$. So in any case we have $l_{\mathfrak{m}}(D_0) \leq 1$. By the Riemann-Roch theorem, we have $i_{\mathfrak{m}}(D_0) \leq \pi$. Applying Lemma 3.2 repeatedly, we can find $P_1, \ldots, P_{i_{\mathfrak{m}}(D_0)}$ in $X-S$ so that $i_{\mathfrak{m}}(D_0+P_1+\cdots+P_{i_{\mathfrak{m}}(D_0)}) = 0$. Choose $P_{i_{\mathfrak{m}}(D_0)+1}, \ldots, P_\pi$ in $X-S$ arbitrarily. We have

$$i_{\mathfrak{m}}(D_0+P_1+\cdots+P_{i_{\mathfrak{m}}(D_0)}) \geq i_{\mathfrak{m}}(D_0+P_1+\cdots+P_{i_{\mathfrak{m}}(D_0)}+P_{i_{\mathfrak{m}}(D_0)+1}+\cdots+P_\pi).$$

(This can be seen by interpreting $i_{\mathfrak{m}}(D)$ as the dimension of the vector space of differential forms ω regular at Q satisfying $(\omega) \geq D$.) So we have $i_{\mathfrak{m}}(D_0+P_1+\cdots+P_\pi) = 0$. By the Riemann-Roch theorem, we have $l_{\mathfrak{m}}(D_0+P_1+\cdots+P_\pi) = 1$. Hence $P_1+\cdots+P_\pi$ is in the set V_1 and V_1 is not empty.

Similarly by Theorem 1.1 (b) applied to the projection $q\colon X \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$ and the invertible sheaf on $X \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D}+p^*(D_0-\mathfrak{m})$, where $p\colon X \times (X-S)^{(\pi)} \to X$ is another projection, we see that the set

$$V_2 = \{D \in (X-S)^{(\pi)} \mid l(D+D_0-\mathfrak{m}) = 0\}$$

is open. Since $\deg(D_0-\mathfrak{m}) < 0$, we have $l(D_0-\mathfrak{m}) = 0$. By the Riemann-Roch theorem, we have $i(D_0-\mathfrak{m}) = \pi$. Applying Lemma 3.2 repeatedly (but taking $\mathfrak{m}=0$), we can find $P_1, \ldots, P_\pi \in X-S$ such that $i(D_0-\mathfrak{m}+P_1+\cdots+P_\pi) = 0$. Then by the Riemann-Roch theorem we have $l(D_0-\mathfrak{m}+P_1+\cdots+P_\pi) = 0$. So $P_1+\cdots+P_\pi$ is in V_2 and V_2 is not empty.

Since $(X-S)^{(\pi)}$ is irreducible, the set $V_{D_0} = V_1 \cap V_2$ is open and non-empty.

LEMMA 3.4. *Fix a point* P_0 *in* S.

(a) *The set*

$$U = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1, \quad l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

(b) *The set*

$$V = \{(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D_2 - D_1 + \pi P_0) = 1, \quad l(D_2 - D_1 + \pi P_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

Proof. (a) Let $p_1, p_2 \colon (X-S)^{(\pi)} \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$ be the projections and let E_i $(i = 1, 2)$ be the pull-backs by $\mathrm{id} \times p_i$ of the universal relative effective Cartier divisor $\mathcal{D}$ on $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$. Put $E = E_1 + E_2$. This is a divisor on $X_{\mathfrak{m}} \times (X - S)^{(\pi)} \times (X - S)^{(\pi)}$.

Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \xrightarrow{\ p\ } & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ (X - S)^{(\pi)} \times (X - S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

By the Riemann-Roch theorem, for any $(D_1, D_2) \in (X - S)^{(\pi)} \times (X - S)^{(\pi)}$, we have

$$l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) \geq \deg(D_1 + D_2 - \pi P_0) + 1 - \pi = 1\,,$$

that is, for any $t \in (X-S)^{(\pi)} \times (X-S)^{(\pi)}$, we have $l_{\mathfrak{m}}(E_t - \pi P_0) \geq 1$. Applying Theorem 1.1 (b) to the projection q and the invertible sheaf corresponding to the divisor $E - p^*(P_0)$, we see that the set

$$U_1 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(E_t - \pi P_0) = 1\}$$

is open. Similarly the set

$$U_2 = \{t \in (X - S)^{(\pi)} \times (X - S)^{(\pi)} \mid l(E_t - \pi P_0 - \mathfrak{m}) = 0\}$$

is also open. Hence the set $U = U_1 \cap U_2$ is open.

Applying Lemma 3.3 to $D_0 = 0$, we see that there exists a $D \in (X - S)^{(\pi)}$ such that $l_{\mathfrak{m}}(D) = 1$ and $l(D - \mathfrak{m}) = 0$. Then $(D, \pi P_0)$ is in U. So U is non-empty. This proves (a).

The proof of (b) is similar and is omitted.

DEFINITION 3.5. A *birational group* over k is a nonsingular variety V together with a rational map $m\colon V\times V\to V$, $(a,b)\mapsto ab$ such that

(a) $(ab)c=a(bc)$ when both sides are defined;

(b) the rational maps $\Phi\colon (a,b)\mapsto(a,ab)$ and $\Psi\colon(a,b)\mapsto(b,ab)$ on $V\times V$ are birational.

PROPOSITION 3.6. *There exists a unique rational map*

$$m\colon (X-S)^{(\pi)}\times(X-S)^{(\pi)}\to(X-S)^{(\pi)}$$

whose domain of definition contains the set U *in* 3.4 (a) *such that* $m(D_1,D_2)$ *is the unique effective divisor that is* $\mathfrak{m}$*-equivalent to* $D_1+D_2-\pi P_0$ *for any* $(D_1,D_2)\in U$*. Moreover* m *makes* $(X-S)^{(\pi)}$ *a birational group.*

Proof. Keep the notations in the proof of Lemma 3.4. Consider the Cartesian squares

$$\begin{array}{ccccccc} X_{\mathfrak{m}}=q^{-1}(t) & \longrightarrow & X_{\mathfrak{m}}\times U & \subset & X_{\mathfrak{m}}\times(X-S)^{(\pi)}\times(X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}}\\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \downarrow\\ \mathrm{spec}(k(t)) & \longrightarrow & U & \subset & (X-S)^{(\pi)}\times(X-S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k).\end{array}$$

Let $\mathcal{L}$ be the restriction to $X_{\mathfrak{m}}\times U$ of the invertible sheaf corresponding to the divisor $E_1+E_2-p^*(\pi P_0)$. By Theorem 1.1 (c) and the choice of U, the sheaf $q_*\mathcal{L}$ is invertible. The canonical homomorphism $q^*q_*\mathcal{L}\to\mathcal{L}$ gives rise to $s\colon\mathcal{O}_{X_{\mathfrak{m}}\times U}\to\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1}$. We claim that the pair $(\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1},s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}}\times U)/U$. According to Remark 2.1, it is enough to check that s is injective and $\mathrm{coker}(s)$ is $\mathcal{O}_U$-flat. Since $\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1}$ is invertible, it is enough to verify s_t is injective for all $t\in U$ by [EGA] §0.10.2.4, where s_t is the homomorphism obtained by restricting s to the fiber of q at t. It suffices to show that the restriction of the canonical homomorphism $q^*q_*\mathcal{L}\to\mathcal{L}$ to the fiber of q at t is injective. By Theorem 1.1 (c) we have $q_*\mathcal{L}\otimes_{\mathcal{O}_U}k(t)=H^0(X_{\mathfrak{m}},\mathcal{L}_t)$. So the restriction of the canonical homomorphism to the fiber is $H^0(X_{\mathfrak{m}},\mathcal{L}_t)\otimes_k\mathcal{O}_{X_{\mathfrak{m}}}\to\mathcal{L}_t$. Denote this map by s'_t; we need to show it is injective. But we have $\dim H^0(X_{\mathfrak{m}},\mathcal{L}_t)=1$ since $t\in U$. If we fix a nonzero element $g\in H^0(X_{\mathfrak{m}},\mathcal{L}_t)$, then s'_t is identified with $\mathcal{O}_{X_{\mathfrak{m}}}\to\mathcal{L}_t$, $a\mapsto ag$. This last map is injective since $X_{\mathfrak{m}}$ is an integral scheme and g can be thought of as a rational function. So s_t is injective. Hence $(\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1},s)$ defines a relative effective Cartier divisor. The restriction of this divisor to the fiber of q at t is the divisor on $X_{\mathfrak{m}}$ defined by the pair $(\mathcal{L}_t,g)$, which is supported on $X_{\mathfrak{m}}-Q$. So the divisor defined by

$(\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1},s)$ is supported on $(X_{\mathfrak{m}}-Q)\times U$. By Proposition 3.1 there exists a unique morphism of varieties $m\colon U\to(X-S)^{(\pi)}$ such that the divisor defined by $(\mathcal{L}\otimes(q^*q_*\mathcal{L})^{-1},s)$ is the pull-back by $\mathrm{id}\times m$ of the universal relative effective Cartier divisor $\mathcal{D}$ on $X_{\mathfrak{m}}\times(X-S)^{(\pi)}$. For any $(D_1,D_2)\in U$, we have $l_{\mathfrak{m}}(D_1+D_2-\pi P_0)=1$ and $l(D_1+D_2-\pi P_0-\mathfrak{m})=0$. So there is one and only one effective divisor $\mathfrak{m}$-equivalent to $D_1+D_2-\pi P_0$ and it is simply $m(D_1,D_2)$.

Similarly, using Lemma 3.4 (b) and Proposition 3.1, one can show that there exists a morphism $r\colon V\to(X-S)^{(\pi)}$ such that $r(D_1,D_2)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_2-D_1+\pi P_0$ for any $(D_1,D_2)\in V$.

Let us verify that m defines a birational group on $(X-S)^{(\pi)}$. First we show

$$m(m(D_1,D_2),D_3)=m(D_1,m(D_2,D_3))$$

when (D_1,D_2), (D_2,D_3), $(m(D_1,D_2),D_3)$ and $(D_1,m(D_2,D_3))$ all belong to U. Indeed $m(m(D_1,D_2),D_3)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $m(D_1,D_2)+D_3-\pi P_0$, and $m(D_1,m(D_2,D_3))$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_1+m(D_2,D_3)-\pi P_0$. But $m(D_1,D_2)+D_3-\pi P_0$ and $D_1+m(D_2,D_3)-\pi P_0$ are $\mathfrak{m}$-equivalent since both are $\mathfrak{m}$-equivalent to $D_1+D_2+D_3-2\pi P_0$. So we have $m(m(D_1,D_2),D_3)=m(D_1,m(D_2,D_3))$.

One can also verify $m(D_1,D_2)=m(D_2,D_1)$ when both (D_1,D_2) and (D_2,D_1) are in U, that is, the operation m is commutative.

Next we show that $\Theta\colon(D_1,D_2)\mapsto(D_1,r(D_1,D_2))$ is the birational inverse of $\Phi\colon(D_1,D_2)\mapsto(D_1,m(D_1,D_2))$ so that Φ is birational. Since the operation m is commutative, the rational map $\Psi\colon(D_1,D_2)\mapsto(D_2,m(D_1,D_2))$ is also birational. Therefore m makes $(X-S)^{(\pi)}$ a birational group.

First we verify $\Phi\Theta(D_1,D_2)=(D_1,D_2)$ whenever the left-hand side is defined. We have

$$\Phi\Theta(D_1,D_2)=\Phi(D_1,r(D_1,D_2))=(D_1,m(D_1,r(D_1,D_2)))\,.$$

Moreover $m(D_1,r(D_1,D_2))$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_1+r(D_1,D_2)-\pi P_0$. But D_2 is also an effective divisor $\mathfrak{m}$-equivalent to $D_1+r(D_1,D_2)-\pi P_0$ since we have

$$D_1+r(D_1,D_2)-\pi P_0\sim_{\mathfrak{m}}D_1+(D_2-D_1+\pi P_0)-\pi P_0=D_2\,.$$

Hence $m(D_1,r(D_1,D_2))=D_2$ and $\Phi\Theta(D_1,D_2)=(D_1,D_2)$.

Similarly one can show that $\Theta\Phi(D_1,D_2)=(D_1,D_2)$ when the left-hand side is defined.

Note that Φ is a regular morphism defined on U and Θ is a regular morphism defined on V. Since

$$\Phi\,\Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta\,\Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps Φ and Θ induce regular morphisms $\Phi\colon U \cap \Phi^{-1}(V) \to V \cap \Theta^{-1}(U)$ and $\Theta\colon V \cap \Theta^{-1}(U) \to U \cap \Phi^{-1}(V)$. To show that Φ and Θ are birational inverses to each other, it is enough to check that $U \cap \Phi^{-1}(V)$ and $V \cap \Theta^{-1}(U)$ are non-empty.

Note that $(D_1, D_2) \in U \cap \Phi^{-1}(V)$ if and only if $(D_1, D_2) \in U$ and

$$l_{\mathfrak{m}}(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - \mathfrak{m}) = 0\,.$$

Since $m(D_1, D_2) \sim_{\mathfrak{m}} D_1 + D_2 - \pi P_0$, the above equations are equivalent to

$$l_{\mathfrak{m}}(D_2) = 1, \quad l(D_2 - \mathfrak{m}) = 0\,.$$

Applying Lemma 3.3 to the divisor $D_0 = 0$, we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D) = 0, \quad l(D - \mathfrak{m}) = 0\}$$

is open and non-empty. Since $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ is irreducible, the set $U \cap ((X - S)^{(\pi)} \times V_0)$ is also open and non-empty. This set is exactly $U \cap \Phi^{-1}(V)$. So $U \cap \Phi^{-1}(V)$ is non-empty.

Similarly $V \cap \Theta^{-1}(U)$ is also non-empty. This completes the proof of the proposition.

4. From birational groups to algebraic groups

Let k be an algebraically closed field, let V be a connected nonsingular variety over k, and let $m\colon V \times V \to V$, $(a, b) \mapsto ab$ be a rational map satisfying $(ab)c = a(bc)$. Assume the rational maps $\Phi(a, b) = (a, ab)$ and $\Psi(a, b) = (b, ab)$ are birational. Then there exist open subsets X_Φ, Y_Φ, X_Ψ and Y_Ψ in $V \times V$ such that Φ induces an isomorphism $X_\Phi \cong Y_\Phi$ and Ψ induces an isomorphism $X_\Psi \cong Y_\Psi$. Put $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$.

It is convenient to write the formulae for Φ^{-1} and Ψ^{-1} as $\Phi^{-1}(a, b) = (a, a^{-1}b)$ and $\Psi^{-1}(a, b) = (ba^{-1}, a)$.

LEMMA 4.1. *Replacing V by an open subset, we may assume the two projections $p_i\colon Z \to V$ $(i = 1, 2)$ are surjective.*

Proof. Note that the two projections $p_i\colon V \times V \to V$, $(i = 1, 2)$ are flat since $V \to \operatorname{spec}(k)$ is flat. So the p_i are open by [EGA] IV, §2.4.6. Hence the $p_i(Z)$ are open. Let $V' = p_1(Z) \cap p_2(Z)$. We claim V' has the property stated in the lemma. Let $C = V - V'$ and let $A = (C \times V) \cup (V \times C)$. The subset $X_\Phi{}'$ of $V' \times V'$ corresponding to X_Φ is the complement in X_Φ of $S = (X_\Phi \cap A) \cup \Phi^{-1}(Y_\Phi \cap A)$. We claim that if the fiber of $p_1 : X_\Phi \to V$ at $v \in V$ is contained in S, then $v \in C$. Thus $p_1 : X_\Phi{}' \to V'$ is surjective.

Let us prove the claim. Assume $(v \times V) \cap X_\Phi \subset S$, but $v \notin C$. We have

$$(v\times V)\cap X_\Phi \subset S \subset A\cup\Phi^{-1}(A) \subset (C\times V)\cup(V\times C)\cup\Phi^{-1}(C\times V)\cup\Phi^{-1}(V\times C).$$

Since V is irreducible, we must have

$$(v \times V) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V), \text{ or } \Phi^{-1}(V \times C).$$

Since $v \notin C$, we have

$$(v \times V) \cap X_\Phi \not\subset C \times V, \quad \Phi^{-1}(C \times V).$$

So

$$(v \times V) \cap X_\Phi \subset V \times C \text{ or } \Phi^{-1}(V \times C).$$

Assume $(v \times V) \cap X_\Phi \subset V \times C$. Note that since $v \notin C$, we have $v \in V'$. Hence $(v \times V) \cap X_\Phi$ is not empty. So we have

$$\begin{aligned}\dim V = \dim((v \times V) \cap X_\Phi) &= \dim(((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\leq \dim(v \times C) < \dim V,\end{aligned}$$

that is, $\dim V < \dim V$. This is impossible.

Assume $(v \times V) \cap X_\Phi \subset \Phi^{-1}(V \times C)$. Then $\Phi((v \times V) \cap X_\Phi) \subset V \times C$. Since Φ is birational, we have

$$\begin{aligned}\dim V = \dim \Phi((v \times V) \cap X_\Phi) &= \dim(\Phi((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\leq \dim(v \times C) < \dim V,\end{aligned}$$

which is again impossible. So we must have $v \in C$.

Next we show that if the fiber of $p_2 : X_\Phi \to V$ at $v \in V$ is contained in S, then $v \in C$, and hence $p_2 : X_\Phi{}' \to V'$ is surjective.

Assume $(V \times v) \cap X_\Phi \subset S$ but $v \notin C$. As before we have

$$(V \times v) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V) \text{ or } \Phi^{-1}(V \times C).$$

Since $v \notin C$, we have $(V \times v) \cap X_\Phi \not\subset V \times C$. By counting dimensions, one can show $(V \times v) \cap X_\Phi \not\subset C \times V$. Since $\Phi^{-1}(C \times V) \subset C \times V$, we have $(V \times v) \cap X_\Phi \not\subset \Phi^{-1}(C \times V)$. So we can only have $(V \times v) \cap X_\Phi \subset \Phi^{-1}(V \times C)$. Then we have a rational map

$$V \xrightarrow{\iota_1} (V \times v) \cap X_\Phi \xrightarrow{\Phi} V \times C \xrightarrow{p_2} C,$$

where $\iota_1(x) = (x, v)$. This map $p_2 \Phi \iota_1 \colon V \to C$ is nothing but $x \mapsto xv$ and it is birational. (Its birational inverse is $p_1 \Psi^{-1} \iota_2$, where $\iota_2(x) = (v, x)$.) So V is birational to C. This is impossible since $\dim V \neq \dim C$. So we must have $v \in C$. This finishes the proof of the surjectivity of $p_2 \colon X_\Phi{}' \to V'$.

Similarly $p_i \colon X'_\Phi, Y_\Phi{}', X'_\Psi, Y'_\Psi \to V'$ are surjective. Since the fibers of $p_i \colon V \times V \to V$ are irreducible, the projection $p_i \colon Z' = X'_\Phi \cap Y'_\Phi \cap X'_\Psi \cap Y'_\Psi \to V'$ is also surjective.

Having replaced V as in Lemma 4.1, we may assume V satisfies the following properties:

PROPERTY 4.2. *There exists an open set $Z \subset V \times V$ such that Φ, Φ^{-1}, Ψ, and Ψ^{-1} are defined on Z, the restrictions $\Phi|_Z$ and $\Psi|_Z$ are open immersions, and the projections $p_i \colon Z \to V$ are surjective. Hence for every $v \in V$, the maps Φ, Φ^{-1}, Ψ and Ψ^{-1} are defined at (v, x) and at (x, v), provided x is generic, i.e. lies in an open set.*

LEMMA 4.3. *Assume 4.2 holds. Denote the closure of the graph of m in $V \times V \times V$ by Γ. Then the projections $p_{ij} \colon \Gamma \to V \times V$ $(1 \leq i < j \leq 3)$ are open immersions.*

Proof. By [EGA] III, §4.4.9, it suffices to show that the maps p_{ij} are set-theoretically injective. Let x be a point of V. The two rational maps $\Gamma \to V$ defined by

$$(a, b, c) \mapsto (xa)b \quad \text{and} \quad (a, b, c) \mapsto xc$$

are equal by the associative law. Let (a, b, c), $(a, b, c') \in \Gamma$. Choose x so that $(xa)b$ is defined and (x, c), $(x, c') \in Z$. Then $xc = (xa)b = xc'$. Hence $\Phi(x, c) = \Phi(x, c')$. Since Φ is an open immersion on Z, we have $(x, c) = (x, c')$. Hence $c = c'$. This shows that $p_{12} \colon \Gamma \to V \times V$ is injective. Similarly one can show the other projections are injective.

We will now expand V to the group we want by glueing translates of V. Let s be a point of V and let V_s be a copy of V thought of as the translate $V_s = \{vs \mid v \in V\}$. The subset $W_s = (V \times s \times V) \cap \Gamma$ is closed in $V \times s \times V \cong V \times V$, and the two projections $W_s \to V$ are open immersions because they are the base extensions of the open immersions $p_{ij}\colon \Gamma \to V \times V$ by the base changes $V \times s \to V \times V$ and $s \times V \to V \times V$, respectively. Therefore W_s defines glueing data and yields a separated scheme $V' = V \cup_{W_s} V_s$.

LEMMA 4.4. *V is an open dense subset of V' and V' satisfies 4.2.*

Proof. Since xs is defined for generic $x \in V$, the set $V \cap V_s$ is not empty. So V' is irreducible and V is dense in V'. We have

$$V' \times V' = (V \times V) \cup (V \times V_s) \cup (V_s \times V) \cup (V_s \times V_s).$$

For every point $v \in V$, denote by v_s the point v considered as a point in V_s. Note that if $(v, s) \in Z$, then $vs \in V$ and $v_s \in V_s$ are glued together in V'. Define $R_s\colon V \to V_s$ by $v \mapsto v_s$. Let

$$W_1 = \{(a,b) \in V \times V \mid (a,b),\quad (s,a) \text{ and } (b, sa^{-1}) \text{ are all in } Z\}.$$

This is a non-empty open subset of Z. Take $Z_1 = (\mathrm{id} \times R_s)(W_1) \subset V \times V_s$. We define Φ, Ψ, Φ^{-1} and Ψ^{-1} on Z_1 by

$$\begin{aligned}
\Phi(a, b_s) &= (a, (ab)_s) \in V \times V_s,\\
\Psi(a, b_s) &= (b_s, (ab)_s) \in V_s \times V_s,\\
\Phi^{-1}(a, b_s) &= (a, (a^{-1}b)_s) \in V \times V_s,\\
\Psi^{-1}(a, b_s) &= (b(sa^{-1}), a) \in V \times V
\end{aligned}$$

for any $(a, b_s) \in Z_1$. Let

$$W_2 = \{(a,b) \in V \times V \mid (a,b),\ (s,b),\ (a,sb),\ (s,a^{-1}b) \text{ and } (bs^{-1}, a) \text{ are all in } Z\}.$$

This is a non-empty open subset of Z. Take $Z_2 = (R_s \times \mathrm{id})(W_2) \subset V_s \times V$. We define Φ, Ψ, Φ^{-1}, and Ψ^{-1} on Z_2 by

$$\begin{aligned}
\Phi(a_s, b) &= (a_s, a(sb)) \in V_s \times V,\\
\Psi(a_s, b) &= (b, a(sb)) \in V \times V,\\
\Phi^{-1}(a_s, b) &= (a_s, s^{-1}(a^{-1}b)) \in V_s \times V,\\
\Psi^{-1}(a_s, b) &= ((bs^{-1})a^{-1}, a_s) \in V \times V_s
\end{aligned}$$

for any $(a_s, b) \in Z_2$.

Let $Z' = Z \cup Z_1 \cup Z_2$. It is an open subset of $V' \times V'$, and Φ, Ψ, Φ^{-1}, Ψ^{-1} are defined on it. One can show that $\Phi|_{Z'}$ and $\Psi|_{Z'}$ are open immersions. Given $v \in V'$, we need to show there exists $x \in V'$ such that (x, v) and (v, x) are in Z'. This is true if $v \in V$ by the property of Z. If $v \in V_s$, then $v = a_s$ for some $a \in V$. We leave it to the reader to show that $(x, a_s) \in Z_1$ and $(a_s, x) \in Z_2$ for generic x in V. This completes the proof of the lemma.

The above lemma allows us to replace V by V', hence to expand V whenever there exists a point s in V such that vs is not defined for all $v \in V$, and we can expand V' if there exists a point $s' \in V'$ such that $v's'$ is not defined for all $v' \in V'$. Denote the result of finitely many such expansions also by V', and let $U \subset V \times V \times V'$ be the closure of Γ. By Lemma 4.3 applied to V', the projection $p_{12}: U \to V \times V$ is an open immersion. Its image is the set of points (a, b) such that $m: V \times V \to V'$ is defined at (a, b). If $V \times s \not\subset p_{12}(U)$ for some point s in V, then replacing V' by $V' \cup V_s{}'$ increases both V' and $p_{12}(U)$. Using noetherian induction on open subschemes of $V \times V$, we may assume that after finitely many expansions, $V \times s \subset p_{12}(U)$ for all points $s \in V$. Then we have $p_{12}(U) = V \times V$.

PROPOSITION 4.5. *Let V, V', and U be as above. If $p_{12}(U) = V \times V$, then the operation $m: V' \times V' \to V'$ is everywhere defined on V' and makes V' an algebraic group.*

Proof. Take (a', b') in $V' \times V'$. Choose a point x so that $a'x$ and $x^{-1}b'$ are both defined and lie in V. Then we can define $m(a', b') = (a'x)(x^{-1}b')$. Similarly one can define $a'^{-1}b'$ and $b'a'^{-1}$. In this way we extend m, Φ, Ψ, Φ^{-1} and Ψ^{-1} to $V' \times V'$. The verification of the group axioms is routine and is omitted.

5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on $(X - S)^{(\pi)}$. The algebraic group associated to this birational group is called the *generalized jacobian* of $X_{\mathfrak{m}}$ and is denoted by $J_{\mathfrak{m}}$. It is a commutative algebraic group.

Let D_0 be a divisor on X prime to S of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1, \quad l(D + D_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)}$. We have the following

LEMMA 5.1. *There exists a unique morphism of varieties*

$$\alpha_{D_0}\colon V_{D_0} \to (X-S)^{(\pi)}$$

such that $\alpha_{D_0}(D)$ *is the unique effective divisor* $\mathfrak{m}$*-equivalent to* $D+D_0$ *for any* $D \in V_{D_0}$. *Moreover* α_{D_0} *is birational.*

Proof. Consider the Cartesian squares

$$\begin{array}{ccccc} X_{\mathfrak{m}} \times V_{D_0} & \subset & X_{\mathfrak{m}} \times (X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow & & \downarrow \\ V_{D_0} & \subset & (X-S)^{(\pi)} & \longrightarrow & \operatorname{spec}(k)\,. \end{array}$$

Let $\mathcal{L}$ be the restriction to $X_{\mathfrak{m}} \times V_{D_0}$ of the invertible sheaf on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ that corresponds to the divisor $\mathcal{D} + p^*(D_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor. By Theorem 1.1 (c) the sheaf $q_*\mathcal{L}$ is invertible. The canonical map $q^*q_*\mathcal{L} \to \mathcal{L}$ induces a homorphism $s\colon \mathcal{O}_{X_{\mathfrak{m}} \times V_{D_0}} \to \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$. Using Remark 2.1, one can show that the pair $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ induces a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times V_{D_0})/V_{D_0}$. Applying Proposition 3.1 to this divisor, one gets the existence of α_{D_0}. For any $D \in V_{D_0}$, we have $l_{\mathfrak{m}}(D+D_0) = 1$ and $l(D+D_0-\mathfrak{m}) = 0$. So there is one and only one effective divisor $\mathfrak{m}$-equivalent to $D+D_0$, and this effective divisor is simply $\alpha_{D_0}(D)$.

We claim that α_{-D_0} is the birational inverse of α_{D_0}. We have

$$\begin{aligned} \alpha_{D_0}^{-1}(V_{-D_0}) &= \{D \mid D \in V_{D_0},\ \alpha_{D_0}(D) \in V_{-D_0}\} \\ &= \{D \mid D \in V_{D_0},\ l_{\mathfrak{m}}(\alpha_{D_0}(D) - D_0) = 1,\ l(\alpha_{D_0}(D) - D_0 - \mathfrak{m}) = 0\} \\ &= V_{D_0} \cap \{D \mid l_{\mathfrak{m}}(D) = 1,\ l(D-\mathfrak{m}) = 0\} \\ &= V_{D_0} \cap V_0\,. \end{aligned}$$

By Lemma 3.3 both V_{D_0} and V_0 are open and non-empty. Since $(X-S)^{(\pi)}$ is irreducible, the set $V_{D_0} \cap V_0$ is also open and non-empty, that is, $\alpha_{D_0}^{-1}(V_{-D_0})$ is open and non-empty. One can easily show that on this open set $\alpha_{-D_0} \circ \alpha_{D_0}$ is defined and is the identity. Similarly one can show $\alpha_{-D_0}^{-1}(V_{D_0})$ is open and non-empty, and on it $\alpha_{D_0} \circ \alpha_{-D_0}$ is defined and is the identity. So α_{D_0} is birational.

We have a birational map $\varphi\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ by the construction of $J_{\mathfrak{m}}$. Let $\operatorname{dom}(\varphi)$ be an open subset of $(X-S)^{(\pi)}$ such that $\varphi|_{\operatorname{dom}(\varphi)}$ is an open immersion, Moreover we may assume that for any $a \in \operatorname{dom}(\varphi)$, both (a, x)

and (x, a) lie in the set U defined in Lemma 3.4 (a) if x is generic, i.e., lies in some open set. In particular, $m(a, x)$ and $m(x, a)$ are defined for generic x.

Let

$$U_{D_0} = V_{D_0} \cap \mathrm{dom}(\varphi) \cap \alpha_{D_0}^{-1}(\mathrm{dom}(\varphi)).$$

Note that U_{D_0} is open and non-empty since $(X - S)^{(\pi)}$ is irreducible and α_{D_0} is birational. Moreover $\varphi(D)$ and $\varphi(\alpha_{D_0}(D))$ are defined for any $D \in U_{D_0}$. Define

$$\theta_0(D_0) = \varphi(\alpha_{D_0}(D)) - \varphi(D).$$

LEMMA 5.2. *$\theta_0(D_0)$ does not depend on the choice of D.*

Proof. Let D_1 and D_2 be two elements in U_{D_0}. We need to show that

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

Choose $D_3 \in U_{D_0}$ so that $(\alpha_{D_0}(D_1), D_3)$, $(D_1, \alpha_{D_0}(D_3))$, $(\alpha_{D_0}(D_2), D_3)$ and $(D_2, \alpha_{D_0}(D_3))$ all lie in the set U defined in Lemma 3.4 (a). Such a D_3 exists. Indeed, if $(\alpha_{D_0}(D_1), x)$, (D_1, x), $(\alpha_{D_0}(D_2), x)$ and (D_2, x) all lie in U for x lying in an open set O, then we may choose D_3 to be any element in $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$. Note that $U_{D_0} \cap O \cap \alpha_{D_0}^{-1}(O)$ is not empty since α_{D_0} is birational and $(X - S)^{(\pi)}$ is irreducible.

We have

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(m(\alpha_{D_0}(D_1), D_3)),$$
$$\varphi(D_1) + \varphi(\alpha_{D_0}(D_3)) = \varphi(m(D_1, \alpha_{D_0}(D_3))).$$

Since

$$m(\alpha_{D_0}(D_1), D_3) \sim_{\mathrm{m}} \alpha_{D_0}(D_1) + D_3 - \pi P_0 \sim_{\mathrm{m}} D_1 + D_0 + D_3 - \pi P_0,$$
$$m(D_1, \alpha_{D_0}(D_3)) \sim_{\mathrm{m}} D_1 + \alpha_{D_0}(D_3) - \pi P_0 \sim_{\mathrm{m}} D_1 + D_3 + D_0 - \pi P_0,$$

we have

$$m(\alpha_{D_0}(D_1), D_3) = m(D_1, \alpha_{D_0}(D_3)).$$

Hence

$$\varphi(\alpha_{D_0}(D_1)) + \varphi(D_3) = \varphi(D_1) + \varphi(\alpha_{D_0}(D_3)),$$

that is,

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Similarly we have

$$\varphi(\alpha_{D_0}(D_2)) - \varphi(D_2) = \varphi(\alpha_{D_0}(D_3)) - \varphi(D_3).$$

Therefore

$$\varphi(\alpha_{D_0}(D_1)) - \varphi(D_1) = \varphi(\alpha_{D_0}(D_2)) - \varphi(D_2).$$

This proves the lemma.

Thus we have a well-defined map $\theta_0\colon \mathrm{Div}^{(0)} \to J_{\mathfrak{m}}$ from the set of divisors of degree 0 on X prime to S to $J_{\mathfrak{m}}$.

LEMMA 5.3. *θ_0 is a homomorphism.*

Proof. Let $D_0, E_0 \in \mathrm{Div}^{(0)}$ and let $F_0 = D_0 + E_0$. Choose $D \in U_{D_0}$, $E \in U_{E_0}$ and $F \in U_{F_0}$ so that

$$\big(\alpha_{D_0}(D), \alpha_{E_0}(E)\big),\ (D,E),\ \big(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F\big) \text{ and } \big(m(D,E), \alpha_{F_0}(F)\big)$$

all lie in the set U defined in Lemma 3.4 (a). We have

$$\alpha_{D_0}(D) + \alpha_{E_0}(E) + F \sim_{\mathfrak{m}} D + D_0 + E + E_0 + F = D + E + F + D_0 + E_0\,,$$
$$D + E + \alpha_{F_0}(F) \sim_{\mathfrak{m}} D + E + F + F_0 = D + E + F + D_0 + E_0\,.$$

So

$$m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F) = m(m(D,E), \alpha_{F_0}(F))\,.$$

Hence

$$\varphi(m(m(\alpha_{D_0}(D), \alpha_{E_0}(E)), F)) = \varphi(m(m(D,E), \alpha_{F_0}(F))\,.$$

Therefore

$$\varphi(\alpha_{D_0}(D)) + \varphi(\alpha_{E_0}(E)) + \varphi(F) = \varphi(D) + \varphi(E) + \varphi(\alpha_{F_0}(F))\,,$$

or equivalently,

$$(\varphi(\alpha_{D_0}(D)) - \varphi(D)) + (\varphi(\alpha_{E_0}(E)) - \varphi(E)) = \varphi(\alpha_{F_0}(F)) - \varphi(F)\,.$$

This last equality is exactly

$$\theta_0(D_0) + \theta_0(E_0) = \theta_0(D_0 + E_0)\,.$$

So θ_0 is a homomorphism.

We define $\theta\colon \mathrm{Div} \to J_{\mathfrak{m}}$ from the group of divisors on X prime to S to $J_{\mathfrak{m}}$ by

$$\theta(D) = \theta_0(D - \deg(D)P_0)\,.$$

Obviously θ is a homomorphism.

PROPOSITION 5.4. *The homomorphism θ is surjective and* $\ker(\theta)$ *consists of divisors* $\mathfrak{m}$*-equivalent to integral multiples of* P_0.

Proof. Assume $\sum_{i=1}^{\pi} P_i$ is in $\operatorname{dom}(\varphi)$. We have

$$\theta\Big(\sum_{i=1}^{\pi} P_i\Big) = \theta_0\Big(\sum_{i=1}^{\pi} P_i - \pi P_0\Big) = \varphi(\alpha_{D_0}(D)) - \varphi(D),$$

where $D_0 = \sum_{i=1}^{\pi} P_i - \pi P_0$ and $D \in U_{D_0}$. We may choose D so that $m(\sum_{i=1}^{\pi} P_i, D)$ is defined and is the unique effective divisor $\mathfrak{m}$-equivalent to $\sum_{i=1}^{\pi} P_i + D - \pi P_0$. Since $\alpha_{D_0}(D)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D + D_0 = D + \sum_{i=1}^{\pi} P_i - \pi P_0$, we have $m(\sum_{i=1}^{\pi} P_i, D) = \alpha_{D_0}(D)$. Hence $\varphi(m(\sum_{i=1}^{\pi} P_i, D)) = \varphi(\alpha_{D_0}(D))$. So $\varphi(\sum_{i=1}^{\pi} P_i) + \varphi(D) = \varphi(\alpha_{D_0}(D))$. Therefore $\varphi(\alpha_{D_0}(D)) - \varphi(D) = \varphi(\sum_{i=1}^{\pi} P_i)$, that is,

$$\theta\Big(\sum_{i=1}^{\pi} P_i\Big) = \varphi\Big(\sum_{i=1}^{\pi} P_i\Big).$$

This is true whenever $\sum_{i=1}^{\pi} P_i$ is in $\operatorname{dom}(\varphi)$.

Since $\varphi|_{\operatorname{dom}(\varphi)}$ is an open immersion, $\varphi(\operatorname{dom}(\varphi))$ is an open subset of $J_{\mathfrak{m}}$. The image of θ contains this open subset. But $J_{\mathfrak{m}}$ is generated by any open subset. So we must have $\operatorname{Im}(\theta) = J_{\mathfrak{m}}$ and θ is surjective.

Assume $E \in \ker(\theta)$. Then $\theta_0(E - \deg(E)P_0) = 0$. Put $E_0 = E - \deg(E)P_0$. Then for any $F \in U_{E_0}$, we have

$$\varphi(\alpha_{E_0}(F)) - \varphi(F) = \theta_0(E - \deg(E)P_0) = 0.$$

Hence $\varphi(\alpha_{E_0}(F)) = \varphi(F)$. But φ is an open immersion on $\operatorname{dom}(\varphi)$. So we have $\alpha_{E_0}(F) = F$. Since $\alpha_{E_0}(F) \sim_{\mathfrak{m}} F + E_0$, we have $F \sim_{\mathfrak{m}} F + E_0$. Hence $E_0 \sim_{\mathfrak{m}} 0$, that is, $E \sim_{\mathfrak{m}} \deg(E)P_0$. So E is $\mathfrak{m}$-equivalent to an integral multiple of P_0.

Conversely assume E is $\mathfrak{m}$-equivalent to an integral multiple of P_0 and let us prove that $\theta(E) = 0$. Again let $E_0 = E - \deg(E)P_0$. Then $E_0 \sim_{\mathfrak{m}} 0$. Choose $F \in U_{E_0} \cap U_0$, where U_0 is the set U_{D_0} defined before by taking $D_0 = 0$. We have

$$\theta(E) = \theta_0(E_0) = \varphi(\alpha_{E_0}(F)) - \varphi(F),$$
$$\theta(0) = \varphi(\alpha_0(F)) - \varphi(F).$$

Note that $F + E_0 \sim_{\mathfrak{m}} F$ since $E_0 \sim_{\mathfrak{m}} 0$. But $\alpha_{E_0}(F)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $F + E_0$, and $\alpha_0(F)$ is the unique effective divisor $\mathfrak{m}$-equivalent to F. So we must have $\alpha_{E_0}(F) = \alpha_0(F)$. Therefore $\theta(E) = \theta(0) = 0$.

Regarding a point P in $X-S$ as a divisor, we can calculate $\theta(P)$. In this way we get a map $\theta\colon X-S \to J_{\mathfrak{m}}$.

PROPOSITION 5.5. *The map $\theta\colon X-S \to J_{\mathfrak{m}}$ is a morphism of algebraic varieties.*

Proof. Let $P \in X-S$ and let $D_0 = P - P_0$. Fix a $D \in U_{D_0}$. Consider the set $W_1 = \{R \in X-S \mid l_{\mathfrak{m}}(D+R-P_0) = 1\}$. By the Riemann-Roch theorem, for any R in $X-S$, we have $l_{\mathfrak{m}}(D+R-P_0) \geq 1$. Applying Theorem 1.1 (b) to the projection $q\colon X_{\mathfrak{m}} \times (X-S) \to X-S$ and the invertible sheaf corresponding to the divisor $\mathcal{D} + p^*(D-P_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor on $X_{\mathfrak{m}} \times (X-S)$ and $p\colon X_{\mathfrak{m}} \times (X-S) \to X_{\mathfrak{m}}$ is another projection, we see that W_1 is open in $X-S$. Similarly one can show $W_2 = \{R \in X-S \mid l(D+R-P_0-\mathfrak{m}) = 0\}$ is also open in $X-S$. So

$$W = W_1 \cap W_2 = \{R \in X-S \mid l_{\mathfrak{m}}(D+R-P_0) = 1, \quad l(D+R-P_0-\mathfrak{m}) = 0\}$$

is open in $X-S$. It is non-empty since $P \in W$ by our choice of D. By Proposition 3.1 we have a morphism $\gamma\colon W \to (X-S)^{(\pi)}$ of algebraic varieties such that for every $R \in W$, $\gamma(R)$ is the unique effective divisor that is $\mathfrak{m}$-equivalent to $D+R-P_0$. Since $\alpha_{R-P_0}(D)$ is the unique effective divisor that is $\mathfrak{m}$-equivalent to $D+R-P_0$, we have $\gamma(R) = \alpha_{R-P_0}(D)$. Replacing W by an open subset containing P, we may assume $\mathrm{Im}(\gamma) \subset \mathrm{dom}(\varphi)$. Note that for any $R \in W$, we have $D \in U_{R-P_0}$, and

$$\theta(R) = \theta_0(R-P_0) = \varphi((\alpha_{R-P_0}(D)) - \varphi(D) = \varphi(\gamma(R)) - \varphi(D),$$

that is, $\theta(R) = \varphi(\gamma(R)) - \varphi(D)$. So $\theta = \varphi \circ \gamma - \varphi(D)$ on W. This proves θ is a morphism of algebraic varieties in an open subset containing P. Since $P \in X-S$ is arbitrary, θ is a morphism of algebraic varieties.

The morphism $\theta\colon X-S \to J_{\mathfrak{m}}$ induces a morphism of algebraic varieties $\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$.

PROPOSITION 5.6. *$\theta\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$ coincides with the birational map $\varphi\colon (X-S)^{(\pi)} \to J_{\mathfrak{m}}$. In particular φ is everywhere defined.*

Proof. Let $\sum_{i=1}^{\pi} P_i \in \mathrm{dom}(\varphi)$. By the proof of Proposition 5.4, we have $\varphi(\sum_{i=1}^{\pi} P_i) = \theta(\sum_{i=1}^{\pi} P_i)$. So $\varphi = \theta$ as rational maps.

Thus there is no difference between φ and θ. From now on we denote the map φ also by θ. We summarize what we have so far in the following theorem.

THEOREM 1. *There is a morphism of algebraic varieties* $\theta\colon X - S \to J_{\mathfrak{m}}$ *satisfying the following properties:*

(a) *The extension of* θ *to the group of divisors on* X *prime to* S *induces, by passing to quotient, an isomorphism between the group* $C_{\mathfrak{m}}^0$ *of classes of divisors of degree zero with respect to* $\mathfrak{m}$*-equivalence and the group* $J_{\mathfrak{m}}$.

(b) *The extension of* θ *to* $(X - S)^{(\pi)}$ *induces a birational map from* $X^{(\pi)}$ *to* $J_{\mathfrak{m}}$.

The following theorem characterizes $J_{\mathfrak{m}}$ by a universal property:

THEOREM 2. *Let* $f\colon X \to G$ *be a rational map from* X *to a commutative algebraic group* G *and assume* $\mathfrak{m}$ *is a modulus for* f. *Then there is a unique homomorphism* $F\colon J_{\mathfrak{m}} \to G$ *of algebraic groups such that* $f = F \circ \theta + f(P_0)$.

Proof. Replacing f by $f - f(P_0)$, we may assume $f(P_0) = 0$. Since $\mathfrak{m}$ is a modulus for f, the extension of f to the group of divisors of X prime to S induces a homomorphism $C_{\mathfrak{m}}^0 \to G$ by passing to quotient. By Theorem 1 (a) we have $J_{\mathfrak{m}} \cong C_{\mathfrak{m}}^0$ as groups. So we have a homomorphism of groups $F\colon J_{\mathfrak{m}} \to G$ such that $f = F\theta$. It remains to prove F is a morphism of algebraic varieties. By Theorem 1 (b) we have a birational map $\theta\colon (X - S)^{(\pi)} \to J_{\mathfrak{m}}$. Denote the extension of f to $(X - S)^{(\pi)}$ by f'. Then $F\theta = f'$. Since θ is birational, it induces an isomorphism between an open subvariety of $(X - S)^{(\pi)}$ and an open subvariety of $J_{\mathfrak{m}}$. Moreover f' is a morphism of algebraic varieties. Hence F is a morphism of algebraic varieties when restricted to some open subset of $J_{\mathfrak{m}}$. The whole $J_{\mathfrak{m}}$ can be obtained from this open subset by translation. So F is a morphism of algebraic varieties.

6. GENERALIZED JACOBIANS AND PICARD SCHEMES

In this section we prove $J_{\mathfrak{m}}$ is the Picard scheme of $X_{\mathfrak{m}}$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

We have $q_*\mathcal{O}_{X_{\mathfrak{m}} \times T} = \mathcal{O}_T$ by [EGA] III, § 1.4.15, the fact $H^0(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}}) = k$, and the fact that $T \to \mathrm{spec}(k)$ is flat. The morphism q has a section $s\colon T \to X_{\mathfrak{m}} \times T$, $t \mapsto (P_0, t)$.

LEMMA 6.1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_\mathfrak{m} \times T$. Assume $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the canonical map* $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \to \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_2)$ *induced by s is bijective.*

Proof. Since $\mathcal{L}_1 \cong \mathcal{L}_2$, it is enough to show that the canonical map $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) \to \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ is bijective. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_\mathfrak{m} \times T}(X_\mathfrak{m} \times T) & \longrightarrow & \mathcal{O}_T(T) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) & \longrightarrow & \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)\,, \end{array}$$

where the horizontal arrows are induced by s. We have

$$\begin{aligned} \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) &\cong \mathrm{Hom}(\mathcal{O}_{X_\mathfrak{m} \times T}, \mathcal{L}_1 \otimes \mathcal{L}_1^{-1}) \\ &\cong \mathrm{Hom}(\mathcal{O}_{X_\mathfrak{m} \times T}, \mathcal{O}_{X_\mathfrak{m} \times T}) \cong \mathcal{O}_{X_\mathfrak{m} \times T}(X_\mathfrak{m} \times T)\,. \end{aligned}$$

Hence the left vertical arrow in the above diagram is bijective. Similarly the right vertical arrow is also bijective. Since $q_*\mathcal{O}_{X_\mathfrak{m} \times T} = \mathcal{O}_T$, we have $\mathcal{O}_{X_\mathfrak{m} \times T}(X_\mathfrak{m} \times T) \cong \mathcal{O}(T)$, and the upper horizontal arrow is bijective. Hence $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_1) \cong \mathrm{Hom}(s^*\mathcal{L}_1, s^*\mathcal{L}_1)$ by the commutativity of the above diagram.

LEMMA 6.2. *Let $\{U_i\}$ be an open covering of T and let $\mathcal{L}_i$ be invertible sheaves on $X_\mathfrak{m} \times U_i$. Assume $s^*\mathcal{L}_i \cong \mathcal{O}_{U_i}$ and $\mathcal{L}_i\,|_{X_\mathfrak{m} \times (U_i \cap U_j)} \cong \mathcal{L}_j\,|_{X_\mathfrak{m} \times (U_i \cap U_j)}$. Then there exists an invertible sheaf $\mathcal{L}$ on $X_\mathfrak{m} \times T$ such that $\mathcal{L}\,|_{X_\mathfrak{m} \times U_i} \cong \mathcal{L}_i$ and $s^*\mathcal{L} \cong \mathcal{O}_T$. Moreover $\mathcal{L}$ is unique up to isomorphism.*

Proof. Fix an isomorphism $\alpha_i \colon s^*\mathcal{L}_i \to \mathcal{O}_{U_i}$ for each i. Let

$$\alpha_{ij} \colon s^*\mathcal{L}_i|_{U_i \cap U_j} \to s^*\mathcal{L}_j|_{U_i \cap U_j}$$

be the isomorphism $(\alpha_j|_{U_i \cap U_j})^{-1} \circ (\alpha_i|_{U_i \cap U_j})$. By Lemma 6.1 the canonical map

$$\mathrm{Hom}(\mathcal{L}_i|_{X_\mathfrak{m} \times (U_i \cap U_j)}, \mathcal{L}_j|_{X_\mathfrak{m} \times (U_i \cap U_j)}) \to \mathrm{Hom}(s^*\mathcal{L}_i|_{U_i \cap U_j}, s^*\mathcal{L}_j|_{U_i \cap U_j})$$

is bijective. So α_{ij} can be lifted uniquely to an isomorphism

$$A_{ij} \colon \mathcal{L}_i\,|_{X_\mathfrak{m} \times (U_i \cap U_j)} \to \mathcal{L}_j\,|_{X_\mathfrak{m} \times (U_i \cap U_j)}\,.$$

By the uniqueness of the lifting and the fact that $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ on $U_i \cap U_j \cap U_k$, we have $A_{jk}A_{ij} = A_{ik}$ on $X_\mathfrak{m} \times (U_i \cap U_j \cap U_k)$. So A_{ij} defines glueing data and we can glue the $\mathcal{L}_i$ together to get an invertible sheaf $\mathcal{L}$ on $X_\mathfrak{m} \times T$. By the construction of $\mathcal{L}$ we have $s^*\mathcal{L} \cong \mathcal{O}_T$. This proves the existence of $\mathcal{L}$. Similarly using Lemma 6.1 one can prove $\mathcal{L}$ is unique up to isomorphism.

LEMMA 6.3. *Assume T is integral. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on $X_m \times T$ satisfying $\mathcal{L}_{1_t} \cong \mathcal{L}_{2_t}$ for all $t \in T$. Then there is an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes q^*\mathcal{M}$.*

Proof. Let $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Then $\mathcal{L}_t \cong \mathcal{O}_{X_m}$. It suffices to show that $\mathcal{L} \cong q^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on T. We have $H^0(X_m, \mathcal{L}_t) = H^0(X_m, \mathcal{O}_{X_m}) = k$. By Theorem 1.1 (c), the sheaf $q_*\mathcal{L}$ is invertible and $q_*\mathcal{L} \otimes k(t) = H^0(X_m, \mathcal{L}_t)$. So the restriction $(q^*q_*\mathcal{L})_t \to \mathcal{L}_t$ of the canonical map $q^*q_*\mathcal{L} \to \mathcal{L}$ to the fiber of q at $t \in T$ is $H^0(X_m, \mathcal{L}_t) \otimes \mathcal{O}_{X_m} \to \mathcal{L}_t$, which is an isomorphism since $\mathcal{L}_t \cong \mathcal{O}_{X_m}$. By Nakayama's Lemma, the canonical map $q^*q_*\mathcal{L} \to L$ is surjective. But since it is a homomorphism of invertible sheaves, it must be bijective. Hence $\mathcal{L} \cong q^*q_*\mathcal{L}$.

Now we use the above lemmas to construct a canonical invertible sheaf on $X_m \times J_m$.

On $X_m \times (X-S)^{(\pi)}$ we have the invertible sheaf corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor and $p\colon X_m \times (X-S)^{(\pi)} \to X_m$ is the projection. Since $\theta\colon (X-S)^{(\pi)} \to J_m$ is birational, there exist open subsets U in $(X-S)^{(\pi)}$ and V in J_m such that θ induces an isomorphism $U \cong V$. Hence we can push-forward the above invertible sheaf on $X_m \times (X-S)^{(\pi)}$ to get an invertible sheaf $\mathcal{L}_V$ on $X_m \times V$. For each $t \in J_m$, denote by $\mathcal{L}(t)$ the invertible sheaf on X_m corresponding to the divisor class in C_m^0 that is mapped to $t \in J_m$ under the canonical isomorphism $C_m^0 \cong J_m$. Obviously the restriction $\mathcal{L}_{V,t}$ of $\mathcal{L}_V$ to the fiber of the projection $q\colon X_m \times J_m \to J_m$ at $t \in V$ is isomorphic to $\mathcal{L}(t)$. The invertible sheaf $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ has the same property, where $s\colon J_m \to X_m \times J_m$ is the section $t \mapsto (P_0, t)$. Thus replacing $\mathcal{L}_V$ by $\mathcal{L}_V \otimes (q^*s^*\mathcal{L}_V)^{-1}$ if necessary, we may assume that $s^*\mathcal{L}_V \cong \mathcal{O}_V$.

For each $a \in J_m$, let $T_{-a}\colon J_m \to J_m$ be the translation $t \mapsto t-a$. Consider the invertible sheaf $\mathcal{L}_{a+V} = (\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a)$ on $X_m \otimes (a+V)$, where $p\colon X_m \times J_m \to X_m$ is the projection. The restriction $\mathcal{L}_{a+V,a+t}$ of $\mathcal{L}_{a+V}$ to the fiber of q at $a+t \in a+V$ is

$$((\mathrm{id} \times T_{-a})^*\mathcal{L}_V \otimes p^*\mathcal{L}(a))_{a+t} = \mathcal{L}_{V,t} \otimes \mathcal{L}(a) = \mathcal{L}(t) \otimes \mathcal{L}(a) = \mathcal{L}(a+t),$$

that is, $\mathcal{L}_{a+V,a+t} = \mathcal{L}(a+t)$. Hence for any $t \in V \cap (a+V)$, we have $\mathcal{L}_{V,t} = \mathcal{L}_{a+V,t}$. By Lemma 6.3, we have

$$\mathcal{L}_V|_{X_m \times (V \cap (a+V))} \cong \mathcal{L}_{a+V}|_{X_m \times (V \cap (a+V))} \otimes q^*\mathcal{M}$$

for some invertible sheaf $\mathcal{M}$ on $V \cap (a+V)$. But since $s^*\mathcal{L}_V \cong \mathcal{O}_V$, we also have $s^*\mathcal{L}_{a+V} = \mathcal{O}_{a+V}$. Hence $\mathcal{M} \cong \mathcal{O}_{V \cap (a+V)}$. Therefore $\mathcal{L}_V|_{X_m \times (V \cap (a+V))} \cong$

$\mathcal{L}_{a+V}|_{X_\mathfrak{m}\times(V\cap(a+V))}$. By Lemma 6.2, we can glue $\mathcal{L}_{a+V}$ $(a \in J_m)$ together to get an invertible sheaf $\mathcal{L}_{J_\mathfrak{m}}$ on $X_\mathfrak{m} \times J_\mathfrak{m}$. It has the property that its restriction to the fiber of q at $t \in J_\mathfrak{m}$ is isomorphic to $\mathcal{L}(t)$ and $s^*\mathcal{L}_{J_\mathfrak{m}} \cong \mathcal{O}_{J_\mathfrak{m}}$.

Define

$$P^0(T) = \{\mathcal{L} \in \operatorname{Pic}(X_\mathfrak{m} \times T) \mid \deg(\mathcal{L}) = 0\}/q^* \operatorname{Pic}(T),$$

where $\deg(\mathcal{L})$ is defined as the leading coefficient of $\chi(\mathcal{L}_t^{\otimes n})$ as a polynomial in n. Since $s^*q^* = \mathrm{id}$, we may define

$$P^0(T) = \{\mathcal{L} \in \operatorname{Pic}(X_\mathfrak{m} \times T) \mid \deg(\mathcal{L}) = 0 \text{ and } s^*\mathcal{L} \cong \mathcal{O}_T\}$$

as well. In particular, we have $\mathcal{L}_{J_\mathfrak{m}} \in P^0(J_\mathfrak{m})$. Using the first definition of $P^0(T)$ and Lemma 6.3, one can show that the pull-back of $\mathcal{L}_{J_\mathfrak{m}}$ by $\mathrm{id} \times \theta \colon X_\mathfrak{m} \times (X-S)^{(\pi)} \to X_\mathfrak{m} \times J_\mathfrak{m}$ is the invertible sheaf on $X_\mathfrak{m} \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} - p^*(\pi P_0)$.

The following theorem says that $J_\mathfrak{m}$ is the Picard scheme of $X_\mathfrak{m}$.

THEOREM 3. *The functor $T \to P^0(T)$ is represented by $J_\mathfrak{m}$. More precisely, for any invertible sheaf $\mathcal{L}$ on $X_\mathfrak{m} \times T$ of degree 0 satisfying $s^*\mathcal{L} \cong \mathcal{O}_T$, there is one and only one morphism of schemes $f \colon T \to J_\mathfrak{m}$ such that $\mathcal{L}$ is the pull-back of $\mathcal{L}_{J_\mathfrak{m}}$ by $\mathrm{id} \times f \colon X_\mathfrak{m} \times T \to X_\mathfrak{m} \times J_\mathfrak{m}$.*

Proof. Let $V_0 = \{D \in (X-S)^{(\pi)} \mid l_\mathfrak{m}(D) = 1, \quad l(D-\mathfrak{m}) = 0\}$. By Lemma 3.3, we know V_0 is non-empty and open in $(X-S)^{(\pi)}$. Note that for every $D \in V_0$, there is one and only one effective divisor in $X_\mathfrak{m}$ that is $\mathfrak{m}$-equivalent to D. Hence the restriction $\theta|_{V_0}$ of $\theta \colon (X-S)^{(\pi)} \to J_\mathfrak{m}$ to V_0 is injective. By [EGA] III, §4.4.9, $\theta|_{V_0}$ is an open immersion.

Consider the Cartesian square

$$\begin{array}{ccc} X_\mathfrak{m} \times T & \xrightarrow{\;p\;} & X_\mathfrak{m} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \operatorname{spec}(k)\,. \end{array}$$

Let $\mathcal{L}' = \mathcal{L} \otimes p^*\mathcal{L}(\pi P_0)$, where $\mathcal{L}(\pi P_0)$ is the invertible sheaf on $X_\mathfrak{m}$ corresponding to the divisor πP_0. Let us prove the theorem under the extra assumption that for every $t \in T$, we have $\dim H^0(X_\mathfrak{m}, \mathcal{L}'_t) = 1$ and $\dim H^0(X, \mathcal{L}'_t \otimes \mathcal{L}(-\mathfrak{m})) = 0$, where $\mathcal{L}(-\mathfrak{m})$ is the invertible sheaf on X corresponding to the divisor $-\mathfrak{m}$. By the Riemann-Roch theorem, for every $t \in T$, we have $\dim H^1(X_\mathfrak{m}, \mathcal{L}'_t) = 0$. By Theorem 1.1 (d) the sheaf $q_*\mathcal{L}'$ is invertible. The canonical map $q^*q_*\mathcal{L}' \to \mathcal{L}'$ induces

$$s\colon \mathcal{O}_{X_{\mathfrak{m}}\times T} \longrightarrow \mathcal{L}' \otimes (q^* q_* \mathcal{L}')^{-1} .$$

Using Remark 2.1, one can show that the pair $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$. By Proposition 3.1, there exists a unique morphism of schemes $g\colon T \longrightarrow (X-S)^{(\pi)}$ such that the pull-back by $\mathrm{id} \times g$ of the universal relative effective Cartier divisor $\mathcal{D}$ is the divisor defined by $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$. Let $f = \theta g$. Then the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f$ is $\mathcal{L}$. This proves the existence of f. To prove f is unique, assume $f\colon T \longrightarrow J_{\mathfrak{m}}$ is a morphism such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f$ is $\mathcal{L}$. By our extra assumption, we must have $\mathrm{Im}(f) \subset \theta(V_0)$. But $\theta|_{V_0}$ is an open immersion. So there exists a morphism $g\colon T \longrightarrow (X-S)^{(\pi)}$ such that $f = \theta g$. We leave it to the reader to prove that the pull-back of the universal relative effective Cartier divisor $\mathcal{D}$ by $\mathrm{id} \times g$ is the divisor defined by the pair $(\mathcal{L}' \otimes (q^*q_*\mathcal{L}')^{-1}, s)$. By Proposition 3.1, such kind of g is unique. So f is also unique.

Now let us prove the theorem. Let t_0 be a point in T. For every point $D \in (X-S)^{(\pi)}$, denote by $\mathcal{L}(D)$ the invertible sheaf on X or on $X_{\mathfrak{m}}$ corresponding to the divisor D. By Lemma 3.3, the set

$$\{D \in (X-S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{t_0} \otimes \mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_{t_0} \otimes \mathcal{L}(D-\mathfrak{m})) = 0\}$$

is non-empty (and open). Fix an element D in this set. Consider the set

$$U_{t_0} = \{t \in T \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t \otimes \mathcal{L}(D)) = 1,\ \dim H^0(X, \mathcal{L}_t \otimes \mathcal{L}(D-\mathfrak{m})) = 0\} .$$

This set is open by the Riemann-Roch theorem and Theorem 1.1 (b). Obviously it contains t_0. So U_{t_0} is an open neighbourhood of t_0. By the theorem with the extra assumption that we have already proved, there exists a unique morphism $f'_{U_{t_0}}\colon U_{t_0} \longrightarrow J_{\mathfrak{m}}$ such that the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by $\mathrm{id} \times f'_{U_{t_0}}$ is $(\mathcal{L} \otimes p^*\mathcal{L}(D - \pi P_0))|_{X_{\mathfrak{m}} \times U_{t_0}}$. Put $f_{U_{t_0}} = f'_{U_{t_0}} + a$, where a is the point in $J_{\mathfrak{m}}$ corresponding to the divisor class $\pi P_0 - D$ in $C^0_{\mathfrak{m}}$. Obviously the pull-back of $\mathcal{L}_{J_{\mathfrak{m}}}$ by the morphism $\mathrm{id} \times f_{U_{t_0}}$ is $\mathcal{L}\,|_{X_{\mathfrak{m}} \times U_{t_0}}$. Moreover, such an $f_{U_{t_0}}$ is unique. So we can glue $f_{U_{t_0}}$ together to get $f\colon T \longrightarrow J_{\mathfrak{m}}$.

APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. *Let A be a commutative ring with identity on which a finite group G acts, let A^G be the invariant subring, and let B be a flat A^G-algebra. Then G acts on $B \otimes_{A^G} A$ through its action on the second factor and the invariant subring of this action is B.*

Proof. We have an exact sequence

$$0 \to A^G \to A \to \prod_{g\in G} A,$$

where $\prod_{g\in G} A$ is the direct product of $|G|$ copies of A, and $A \to \prod_{g\in G} A$ is defined by $a \mapsto (ga - a)$. Since B is a flat A^G-algebra, the tensor product of B with the above sequence remains exact, that is, the sequence

$$0 \to B \to B \otimes_{A^G} A \to \prod_{g\in G} B \otimes_{A^G} A$$

is exact. Hence $B = (B \otimes_{A^G} A)^G$.

Let A be a finitely generated k-algebra on which a finite group G acts. Then A is finite over A^G. For every prime ideal $\mathfrak{q}$ of A^G, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the prime ideals of A lying over $\mathfrak{q}$. It is known that G acts transitively on $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Fix a $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$ be the decomposition group at $\mathfrak{p}$.

LEMMA A.2. *Notation as above. Let $\widehat{A^G{}_{\mathfrak{q}}}$ be the completion of the local ring $A^G{}_{\mathfrak{q}}$ and let $\widehat{A_{\mathfrak{p}}}$ be the completion of the local ring $A_{\mathfrak{p}}$. Then G_d acts on $\widehat{A_{\mathfrak{p}}}$ and $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G{}_{\mathfrak{q}}}$.*

Proof. Since $A^G{}_{\mathfrak{q}}$ is a flat A^G-algebra, we have $(A^G{}_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G{}_{\mathfrak{q}}$ by Lemma A.1. Replacing A by $A^G{}_{\mathfrak{q}} \otimes_{A^G} A$ if necessary, we may thus assume that A^G is a local ring and $\mathfrak{q}$ is the maximal ideal of A^G.

Let $\widehat{A}$ be the completion of A with respect to the $\mathfrak{q}A$-adic topology. Since A is a finite A^G-algebra, we have $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$. On the other hand, we have $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$. Since $\widehat{A^G}$ is a flat A^G-algebra, we have $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$ by Lemma A.1. So we have $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$. Obviously $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$. Therefore $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G}$.

LEMMA A.3. *Let A be a noetherian local ring, let I_i $(i=1,\dots,n)$ be some ideals of A, and let K_i be the kernel of the canonical homomorphism $\widehat{A}\to\widehat{A/I_i}$. If $I=I_1\cdots I_n$, then the kernel of $\widehat{A}\to\widehat{A/I}$ is $K_1\cdots K_n$.*

Proof. Since A is noetherian, we have $\ker(\widehat{A}\to\widehat{A/I_i})=\widehat{I_i}=I_i\widehat{A}$, that is $K_i=I_i\widehat{A}$. Similarly we have $\ker(\widehat{A}\to\widehat{A/I})=I\widehat{A}=I_1\cdots I_n\widehat{A}$. So $\ker(\widehat{A}\to\widehat{A/I})=K_1\cdots K_n$.

Let T be a k-scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathrm{m}}\times T & \longrightarrow & X_{\mathrm{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

We have the following

LEMMA A.4. *Let $s\colon T\to X_{\mathrm{m}}\times T$ be a section of q. Then s is a closed immersion and the closed subscheme D defined by s is a relative effective Cartier divisor on $X_{\mathrm{m}}\times T/T$.*

Proof. Since qs=id is a closed immersion and since q is separated, s is also a closed immersion. The closed subscheme D defined by s is flat because qs=id. Let $\mathcal{I}$ be the sheaf of $\mathcal{O}_{X_{\mathrm{m}}\times T}$-ideals defining D. We have an exact sequence

$$0\to\mathcal{I}\to\mathcal{O}_{X_{\mathrm{m}}\times T}\to\mathcal{O}_D\to 0\,.$$

For any $t\in T$, since $\mathcal{O}_D$ is $\mathcal{O}_T$ flat, the following sequence is exact:

$$0\to\mathcal{I}\otimes_{\mathcal{O}_T}k(t)\to\mathcal{O}_{X_{\mathrm{m}}\times T}\otimes_{\mathcal{O}_T}k(t)\to\mathcal{O}_{D_t}\to 0\,,$$

where D_t is the fiber of $D\to T$ at t. Hence $\mathcal{I}\otimes_{\mathcal{O}_T}k(t)$ is the ideal defining the closed subscheme D_t of $q^{-1}(t)\cong X_{\mathrm{m}}$. Obviously D_t defines a divisor of X_{m}. So for every point $x\in q^{-1}(t)$, the ideal $\mathcal{I}_x\otimes_{\mathcal{O}_T}k(t)$ of $\mathcal{O}_{X_{\mathrm{m}}\times T,x}\otimes_{\mathcal{O}_{T,t}}k(t)$ is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X_{\mathrm{m}}\times T,x}$ is generated by one element whose image in $\mathcal{O}_{X_{\mathrm{m}}\times T,x}\otimes_{\mathcal{O}_{T,t}}k(t)$ is not a zero divisor. By Lemma 2.3, D is a relative effective Cartier divisor.

Consider the sections

$$s_i\colon (X-S)^n \longrightarrow X_{\mathfrak{m}} \times (X-S)^n, \quad (P_1,\dots,P_n) \mapsto (P_i,P_1,\dots,P_n).$$

Denote the relative effective Cartier divisors defined by s_i also by s_i, and let $D = s_1+\dots+s_n$. The relative effective Cartier divisor D can also be regarded as a closed subscheme of $X_{\mathfrak{m}} \times (X-S)^n$. The n-th symmetric group $\mathfrak{S}_n$ acts on $(X-S)^n$ by permuting the factors. It acts on $X_{\mathfrak{m}} \times (X-S)^n$ through its action on the second factor. Obviously D is stable under this action. Let $\mathcal{D}$ be the quotient of D by $\mathfrak{S}_n$.

PROPOSITION A.5.

(a) *The quotient of* $X_{\mathfrak{m}} \times (X-S)^n$ *by* $\mathfrak{S}_n$ *is* $X_{\mathfrak{m}} \times (X-S)^{(n)}$.

(b) *The closed immersion* $D \to X_{\mathfrak{m}} \times (X-S)^n$ *induces a closed immersion* $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$ *and* $\mathcal{D}$ *is a relative effective Cartier divisor on* $(X_{\mathfrak{m}} \times (X-S)^{(n)})/(X-S)^{(n)}$. *Moreover* D *is the pull-back of* $\mathcal{D}$.

Proof. (a) We have a Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X-S)^n & \longrightarrow & X_{\mathfrak{m}} \times (X-S)^{(n)} \\ \downarrow & & \downarrow \\ (X-S)^n & \longrightarrow & (X-S)^{(n)}\,. \end{array}$$

The morphism $X_{\mathfrak{m}} \times (X-S)^{(n)} \to (X-S)^{(n)}$ is flat since it is obtained from the flat morphism $X_{\mathfrak{m}} \to \operatorname{spec}(k)$ through the base extension $(X-S)^{(n)} \to \operatorname{spec}(k)$. Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}} \times (X-S)^n & \longrightarrow & X_{\mathfrak{m}} \times (X-S)^{(n)} \\ \downarrow & & \downarrow \\ (X-S)^n & \longrightarrow & (X-S)^{(n)}\,. \end{array}$$

One can easily show that $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$ is a finite morphism and induces a homeomorphism of $\mathcal{D}$ with a closed subset of $X_{\mathfrak{m}} \times (X-S)^{(n)}$. We are going to show that for any point $y \in \mathcal{D}$, the homomorphism $\mathcal{O}_{X_{\mathfrak{m}} \times (X-S)^{(n)},y} \to \mathcal{O}_{\mathcal{D},y}$ is surjective and the homomorphism $\mathcal{O}_{(X-S)^{(n)},t} \to \mathcal{O}_{\mathcal{D},y}$ is flat, where t is the image of y in $(X-S)^{(n)}$. If this is done, then $\mathcal{D} \to X_{\mathfrak{m}} \times (X-S)^{(n)}$

is a closed immersion and $\mathcal{D} \to (X-S)^{(n)}$ is flat. Obviously the fibers of $\mathcal{D} \to (X-S)^{(n)}$ are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that $\mathcal{D}$ is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_{\mathrm{m}} \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)},y} .$$

Note that $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)},y}$ is a faithfully flat $\mathcal{O}_{X_{\mathrm{m}} \times (X-S)^{(n)},y}$-algebra. Thus to show that $\mathcal{O}_{X_{\mathrm{m}} \times (X-S)^{(n)},y} \to \mathcal{O}_{\mathcal{D},y}$ is surjective, it is enough to show that $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)},y} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective; and to show that $\mathcal{O}_{(X-S)^{(n)},t} \to \mathcal{O}_{\mathcal{D},y}$ is flat, it is enough to show that $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat.

Assume $t = n_1P_1 + \cdots + n_lP_l \in (X-S)^{(n)}$, where the P_i are distinct points of $X-S$, $n_i > 0$ and $\sum_i n_i = n$. Then $y = (P_{i_0}, t) \in X_{\mathrm{m}} \times (X-S)^{(n)}$ for some $i_0 \in \{1, \ldots, l\}$. Let $t' = (P_1, \ldots, P_1, \ldots, P_l, \ldots, P_l) \in (X-S)^n$, where the first n_1 components of t' are P_1, … , and the last n_l components are P_l. The point t' is a point in $(X-S)^n$ lying over $t \in (X-S)^{(n)}$. Let y' be the point (P_{i_0}, t') in $X_{\mathrm{m}} \times (X-S)^n$. It lies over y. Note that y' is also a point in D. With respect to the actions of $\mathfrak{S}_n$ on $(X-S)^n$, on $X_{\mathrm{m}} \times (X-S)^n$, and on D, the decomposition groups at $t' \in (X-S)^n$, at $y' \in X_{\mathrm{m}} \times (X-S)^n$, and at $y' \in D$ are all $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]] ,$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{(X-S)^n,t}$ by permuting $x_{i1}, \ldots, x_{in_i}$ for each i. We have

$$\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^n,y'} \cong k[[x, x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]] ,$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^n,y'}$ by fixing x and permuting $x_{i1}, \ldots, x_{in_i}$ for each i.

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \ldots, n_1 + \cdots + n_{i_0}\}$, the section

$$s_i \colon (X-S)^n \to X_{\mathrm{m}} \times (X-S)^n, \quad (P_1, \ldots, P_n) \mapsto (P_i, P_1, \ldots, P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^n,y'} \to \widehat{\mathcal{O}}_{(X-S)^n,t'} .$$

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^n,y'} \cong k[[x, x_{11}, \ldots, x_{1n_1}, \ldots, x_{l1}, \ldots, x_{ln_l}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n,t'} \cong k[[x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]],$$

this homomorphism induced by s_i is

$$k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]] \to k[[x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]],$$
$$x \mapsto x_{i_0 j},\ x_{\alpha\beta} \mapsto x_{\alpha\beta} \quad (\alpha = 1,\dots,l,\ \beta = 1,\dots,n_\alpha),$$

where $j \in \{1,\dots,n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0-1} + j = i$. The kernel of this homomorphism is the ideal $(x - x_{i_0 j})$. By Lemma A.3, the kernel of the homomorphism $\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^n,y'} \to \widehat{\mathcal{O}}_{D,y'}$ is identified with the ideal $\left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right)$ through the isomorphism

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^n,y'} \cong k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]].$$

Hence

$$\widehat{\mathcal{O}}_{D,y'} \cong k[[x,x_{11},\dots,x_{1n_1},\dots,x_{l1},\dots,x_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right),$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{D,y'}$ by fixing x and permuting $x_{i1},\dots,x_{in_i}$ for each i. Let $\sigma_{i1},\dots,\sigma_{in_i}$ be the elementary symmetric functions in $x_{i1},\dots,x_{in_i}$. By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong k[[x,\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}}(x - x_{i_0 j})\right),$$

$$\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^{(n)},y} \cong k[[x,\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \cong k[[\sigma_{11},\dots,\sigma_{1n_1},\dots,\sigma_{l1},\dots,\sigma_{ln_l}]].$$

Now it is easy to see that $\widehat{\mathcal{O}}_{X_{\mathfrak{m}}\times(X-S)^{(n)},y} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective and $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \to \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat. This proves $\mathcal{D}$ is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D,y'} = \widehat{\mathcal{O}}_{\mathcal{D},y} \widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(n)},t}} \widehat{\mathcal{O}}_{(X-S)^n,t'}.$$

This implies that $D = \mathcal{D} \times_{(X-S)^{(n)}} (X-S)^n$, that is, D is the pull-back of $\mathcal{D}$. This completes the proof of the proposition.

The relative effective Cartier divisor $\mathcal{D}$ is the universal relative effective Cartier divisor.

LEMMA A.6. *Let T be a k-scheme and let $s_i\colon T \to X_{\mathfrak{m}} \times T$ $(i = 1, \ldots, n)$ be some sections of the projection $q\colon X_{\mathfrak{m}} \times T \to T$. Assume the images of s_i lie in $(X_{\mathfrak{m}} - Q) \times T$. Then there is a unique morphism of schemes $f\colon T \to (X - S)^{(n)}$ such that the pull-back by $\mathrm{id} \times f$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_{\mathfrak{m}} \times T$ is $s_1 + \cdots + s_n$.*

Proof. Let $p\colon X_{\mathfrak{m}} \times T \to X_{\mathfrak{m}}$ be the projection. The morphisms $ps_i\colon T \to X_{\mathfrak{m}}$ induce $(ps_1, \ldots, ps_n)\colon T \to X_{\mathfrak{m}}^n$. Since the images of s_i lie in $(X_{\mathfrak{m}} - Q) \times T$, we actually get a morphism $(ps_1, \ldots, ps_n)\colon T \to (X - S)^n$. Composing with the canonical morphism $(X - S)^n \to (X - S)^{(n)}$, we get $f\colon T \to (X - S)^{(n)}$ so that the pull-back of $\mathcal{D}$ by $\mathrm{id} \times f$ is $s_1 + \cdots + s_n$. This proves the existence of f.

To prove the uniqueness of f, we first note that $f\colon T \to (X - S)^{(n)}$ is uniquely determined as a map on the underlying topological space. Indeed, for every point $t \in T$, $f(t)$ is necessarily the point in $(X - S)^{(n)}$ corresponding to the effective divisor $(s_1 + \cdots + s_n)_t$ on $q^{-1}(t) = X_m$. To prove f is unique as a morphism of schemes, it is enough to prove that the homomorphism on local rings $\mathcal{O}_{(X-S)^{(n)}, f(t)} \to \mathcal{O}_{T,t}$ induced by f is uniquely determined. It suffices to prove that $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \to \widehat{\mathcal{O}}_{T,t}$ is uniquely determined.

Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ f\ } & (X - S)^{(n)}\ , \end{array}$$

where D is the closed subscheme of $X_{\mathfrak{m}} \times T$ corresponding to the divisor $s_1 + \cdots + s_n$. Let $A = \widehat{\mathcal{O}}_{T,t}$, let $z \in D$ be a point lying over $t \in T$, and let $y \in \mathcal{D}$ be the image of z. We have $\widehat{\mathcal{O}}_{X_{\mathfrak{m}} \times T, z} \cong A[[x]]$.

Without loss of generality, assume

$$ps_1(t) = \cdots = ps_{n_1}(t) = P_1\,,$$

$$ps_{n_1+1}(t) = \cdots = ps_{n_1+n_2}(t) = P_2\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$ps_{n_1+\cdots+n_{l-1}+1}(t) = \cdots = ps_{n_1+\cdots+n_l}(t) = P_l\,,$$

where $n_i > 0$ $(i = 1, \ldots, l)$, $n_1 + \cdots + n_l = n$, and the P_i are distinct points in $X - S$. Then we have $z = (P_{i_0}, t) \in X_{\mathfrak{m}} \times T$ for some $i_0 \in \{1, \ldots, l\}$.

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$, the section s_i induces a homomorphism $\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times T, z} \to \widehat{\mathcal{O}}_{T,t}$, i.e., $A[[x]] \to A$. Denote the image of x under this homomorphism by $a_{i_0 j}$, where $j \in \{1, \dots, n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0-1} + j = i$. Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D,z} \cong A[[x]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right).$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{D,y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right).$$

$$\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

We have a commutative diagram

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_{D,z} & \longleftarrow & \widehat{\mathcal{O}}_{D,y} \\
\uparrow & & \uparrow \\
\widehat{\mathcal{O}}_{X_{\mathrm{m}} \times T, z} & \longleftarrow & \widehat{\mathcal{O}}_{X_{\mathrm{m}} \times (X-S)^{(n)}, y} \\
\uparrow & & \uparrow \\
\widehat{\mathcal{O}}_{T,t} & \longleftarrow & \widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)}
\end{array}.$$

It is isomorphic to

$$\begin{array}{ccc}
A[[x]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \Big/ \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right) \\
\uparrow & & \uparrow \\
A[[x]] & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \\
\uparrow & & \uparrow \\
A & \longleftarrow & k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]]
\end{array}.$$

In order for this last diagram to commute, it is necessary that $\prod_{j=1}^{n_{i_0}}(x-x_{i_0j})$ be mapped to $\prod_{j=1}^{n_{i_0}}(x-a_{i_0j})$ under the homomorphism

$$k[[x,\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{n_l}]]\to A[[x]]\,.$$

So the image of σ_{i_0j} under the homomorphism

$$k[[\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]]\to A$$

is necessarily the value at $(a_{i_0}1,\ldots,a_{i_0n_{i_0}})$ of σ_{i_0j} considered as a function on $A^{n_{i_0}}$. We see that this is true for any indices i_0 and j if we let z go over the points in D above t. Therefore the homomorphism $k[[\sigma_{11},\ldots,\sigma_{1n_1},\ldots,\sigma_{l1},\ldots,\sigma_{ln_l}]]\to A$ is uniquely determined, that is, the homomorphism $\widehat{\mathcal{O}}_{(X-S)^{(n)},f(t)}\to\widehat{\mathcal{O}}_{T,t}$ is uniquely determined. This concludes the proof of the lemma.

LEMMA A.7. *Let T be a k-scheme and let D be a relative effective Cartier divisor on $(X_{\mathfrak{m}}\times T)/T$ supported on $(X_{\mathfrak{m}}-Q)\times T$ with degree n. Then there exist a flat morphism $T'\to T$ and sections $s_i\colon T'\to X_{\mathfrak{m}}\times T'$ $(i=1,\ldots,n)$ of the projection $X_{\mathfrak{m}}\times T'\to T'$ such that the pull-back of D to $X_{\mathfrak{m}}\times T'$ is equal to $s_1+\cdots+s_n$.*

Proof. By the definition of relative effective Cartier divisors, D is flat over T. On the other hand, $D\to T$ is proper and has finite fibers. So D is finite over T by [EGA] III, §4.4.2. Take $T_1=D$. Then we have a finite flat morphism $T_1\to T$. Consider the commutative diagram

$$\begin{array}{ccccc}
D\times_T T_1 & \xrightarrow{p'} & D & & \\
{\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow & & \\
X_{\mathfrak{m}}\times T_1 & \xrightarrow{p} & X_{\mathfrak{m}}\times T & \longrightarrow & X_{\mathfrak{m}} \\
{\scriptstyle q'}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow \\
D=T_1 & \xrightarrow{qi} & T & \longrightarrow & \mathrm{spec}(k)\,.
\end{array}$$

Let $\Delta\colon D\to D\times_T D=D\times_T T_1$ be the diagonal map. It is a closed immersion since the morphism qi is separated. Take $s_1=i'\Delta$. This is a section of q'. Hence it defines a relative effective Cartier divisor on $(X_{\mathfrak{m}}\times T_1)/T_1$. The pull-back D_1 of the relative effective Cartier divisor D to $X_{\mathfrak{m}}\times T_1$ is the closed subscheme defined by i'. Let $\mathcal{I}_{D_1}$ and $\mathcal{I}_s$ be the ideal sheaves of the closed subschemes defined by i' and s_1, respectively. Since s_1 factors through i', we have $\mathcal{I}_{D_1} \subset \mathcal{I}_s$. Hence $D_1 - s$ is a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T_1)/T_1$ by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor $D_1{}'$ such that $D_1 = s_1 + D_1{}'$. Now we take $T_2 = D_1{}'$. We then have a finite flat morphism $T_2 \to T_1$, a section $s_2 \colon T_2 \to X_{\mathfrak{m}} \times T_2$ of the projection $X_{\mathfrak{m}} \times T_2 \to T_2$, and a relative effective Cartier divisor $D_2{}'$ on $(X_{\mathfrak{m}} \times T_2)/T_2$ such that the pull-back of $D_1{}'$ to $X_{\mathfrak{m}} \times T_2$ is equal to $s_2 + D_2{}'$. Then we take $T_3 = D_2{}', \ldots$. In this way we get finite flat morphisms $T_i \to T_{i-1}$ $(i = 1, \ldots, n)$, sections $s_i \colon T_i \to X_{\mathfrak{m}} \times T_i$, such that the pull-back of D to $X_{\mathfrak{m}} \times T_n$ is equal to $s_1 + \cdots + s_n$, where the s_i denote the relative effective Cartier divisors on $(X_{\mathfrak{m}} \times T_n)/T_n$ induced by the sections s_i. This proves our lemma.

Finally we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma A.7, there exist a finite flat morphism $\pi \colon T' \to T$ and sections $s_i \colon T' \to X_{\mathfrak{m}} \times T'$ $(i = 1, \ldots, n)$ of the projection $X_{\mathfrak{m}} \times T' \to T'$ such that the pull-back $\pi^* D$ of D to $X_{\mathfrak{m}} \times T'$ is equal to $s_1 + \cdots + s_n$. By Lemma A.6, there exists a unique morphism of schemes $f' \colon T' \to (X - S)^{(n)}$ such that the pull-back $f'^* \mathcal{D}$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_{\mathfrak{m}} \times T'$ is $s_1 + \cdots + s_n$. Let $p_1, p_2 \colon T' \times_T T' \to T'$ be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^* f'^* \mathcal{D} = p_1^*(s_1 + \cdots + s_n) = p_1^* \pi^* D = p_2^* \pi^* D = \ldots = (f'p_2)^*(\mathcal{D}).$$

that is, $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$. By Lemma A.6 we have $f'p_1 = f'p_2$. By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes $f \colon T \to (X_{\mathfrak{m}} - Q)^{(n)}$ such that $f' = f\pi$, and the pull-back of $\mathcal{D}$ to $X_{\mathfrak{m}} \times T$ is D.

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