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Addition in $K^*(X,G)$ is given by disjoint union of K-cocycles. Further,

$$K^*(X,G) = K^0(X,G) \oplus K^1(X,G),$$

where $K^i(X,G)$ is the subgroup of $K^*(X,G)$ determined by all K-cocycles (Z,ξ,f) with $\xi \in V^i_G(T^*Z \oplus f^*T^*X)$. The natural homomorphism of abelian groups

$$K^i(X,G) \to K_i[C_0(X) \rtimes G]$$

is defined by

$$(Z,\xi,f) \mapsto \mu(Z,\xi,f).$$

CONJECTURE. *For any G-manifold X, $\mu\colon K^i(X,G) \to K_i[C_0(X) \rtimes G]$ is an isomorphism.*

This conjecture is known to be true if X is a proper G-manifold. If X is proper there is a commutative diagram

$$\begin{array}{ccc} K^*(X,G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ {\scriptstyle i_t}\searrow & & \swarrow{\scriptstyle \alpha} \\ & K^*_G(X) & \end{array}$$

in which each arrow is an isomorphism. $i_t\colon K^*(X,G) \to K^*_G(X)$ maps a K-cocycle (Z,ξ,f) to its topological index, and $\alpha \circ \mu\colon K^*(X,G) \to K^*_G(X)$ maps a K-cocycle (Z,ξ,f) to its analytic index. If G is compact then any G-manifold is proper and commutativity of the diagram is equivalent to the Atiyah-Singer index theorems of [6], [7], [8].

3. HOMOTOPY QUOTIENT

Let W be a topological space. $V^0(W)$ denotes the collection of all complex vector bundles (E_0, E_1, σ) on W with compact support. Thus E_0, E_1 are complex vector bundles on W and $\sigma\colon E_0 \to E_1$ is a morphism of complex vector bundles with Support(σ) compact, where

$$\text{Support}(\sigma) = \{p \in W \mid \sigma\colon E_{0p} \to E_{1p} \text{ is not an isomorphism}\}.$$

Also $V^1(W) = V^0(W \times \mathbf{R})$.

Suppose given an $\mathbf{R}$-vector bundle F on W. Following [9], a *twisted by F K-cycle* on W is a triple (M,ξ,ϕ) such that:

(1) M is a C^∞-manifold without boundary;

(2) $\phi: M \to W$ is a continuous map from M to W;

(3) $\xi \in V^*(T^*M \oplus \phi^*F)$.

As in [9] an equivalence relation is imposed on these twisted by F K-cycles to obtain the twisted by F K-homology of W:

$$K_*^F(W) = K_0^F(W) \oplus K_1^F(W).$$

$K_1^F(W)$ is the subgroup determined by all (M, ξ, ϕ) with $\xi \in V^i(T^*M \oplus \phi^*F)$. If F has a Spin^c-structure then $K_*^F(W)$ is isomorphic to $K_*(W)$, the K-homology of W.

With G as in §2 above, let EG be a contractible space on which G acts freely

$$EG \times G \to EG.$$

Given a G-manifold X, let G act on $EG \times X$ by

$$(p, x)\, g = (pg, xg)$$

($p \in EG$, $x \in X$, $g \in G$). The quotient space $[EG \times X]/G$ will be referred to as the homotopy quotient. Since T^*X is a G-vector bundle on X, the quotient $[EG \times T^*X]/G$ is a vector bundle on $[EG \times X]/G$. Denote this vector bundle by τ and consider the twisted by τ K-homology $K_*^\tau([EG \times X]/G)$. There is a map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G).$$

This map is not quite canonical. First an orientation must be chosen for the Lie algebra of G, so assume that such an orientation has been chosen.

Let (M, ξ, ϕ) be a twisted by τ K-cycle on $[EG \times X]/G$. Now $EG \times X$ is the total space of a principal G-bundle over $[EG \times X]/G$ and this principal bundle can be pulled back via ϕ to yield a principal bundle Z over M

$$\begin{array}{ccc} EG \times X & \xleftarrow{\ \tilde{\phi}\ } & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \xleftarrow[\ \phi\ ]{} & M\,. \end{array}$$

Let $\pi: EG \times X \to X$ be the projection and set $f = \pi \circ \tilde{\phi}$,

$$f: Z \to X.$$

$\xi \in V^*(T^*M \oplus \phi^*\tau)$ lifts to give $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$. Denote the bundle along the fibres of $\rho: Z \to M$ by F. This is a trivial vector bundle since, for each $z \in Z$, F_z is canonically isomorphic to the Lie algebra of G. Using the orientation of this Lie algebra, F has a G-invariant Spin^c-structure so that $\tilde{\xi} \in V_G^*(\rho^* T^* M \oplus f^* T^* X)$ determines $\eta \in V_G^*(F \oplus \rho^* T^* M \oplus f^* T^* X)$. Now $F \oplus \rho^* T^* M = T^* Z$, so (Z, η, f) is a K-cocycle for (X, G). The map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G)$$

is:

$$(M, \xi, \phi) \mapsto (Z, \eta, f)\,.$$

This map has a dimension-shift in it. Set $\epsilon = \dim(G)$. Then with addition of indices mod 2 this map takes $K_i^\tau([EG \times X]/G)$ to $K^{i+\epsilon}(X, G)$.

LEMMA 1. *If G is torsion free then $K_*^\tau([EG \times X]/G) \to K^*(X, G)$ is an isomorphism.*

Proof. Let (Z, ξ, f) be a K-cocycle for (X, G). The action of G on Z is proper, so each isotropy group is compact. Since G is assumed to be torsion free this implies that the action of G on Z is free. Hence Z is a G-principal bundle over G/Z, and thus Z maps equivariantly to EG. Combining this with $f: Z \to X$ we obtain a commutative diagram

$$\begin{array}{ccc} EG \times X & \longleftarrow & Z \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ {[EG \times X]} & \longleftarrow & Z/G\,. \end{array}$$

Denote the map of Z/G to $[EG \times X]/G$ by ϕ. Then $\xi \in V_G^*(T^* Z \oplus f^* T^* X)$ determines $\xi' \in V_G^*(\rho^* T^*(Z/G) \oplus f^* T^* X)$. Since the action of G on Z is free ξ' descends to give $\theta \in V^*(T^*(Z/G) \oplus \tau)$. Then

$$(Z, \xi, f) \to (Z/G, \theta, \phi)$$

maps $K^*(X, G)$ to $K_*^\tau([EG \times X]/G)$ and provides an inverse to the map $K_*^\tau([EG \times X]/G) \to K^*(X, G)$. $\square$

REMARK 2. If G is the trivial one-element group then the isomorphism of the lemma becomes

$$K_*^{T^* X}(X) \cong K^*(X)\,.$$

If X is a Spin^c-manifold then $K_*^{T^* X}(X) \cong K_*(X)$, so that in this case the isomorphism of the lemma becomes the Poincaré duality isomorphism $K_*(X) \cong K^*(X)$.

When G has torsion, the map $K_*^T([EG \times X]/G) \to K^*(X,G)$ can fail to be an isomorphism. The simplest example of this is obtained by taking X to be a point and $G = \mathbf{Z}/2\mathbf{Z}$.

When G has torsion, $K_*^T([EG \times X]/G)$ appears to be only a first approximation to $K^*(X,G)$ and $K_*[C_0(X) \rtimes G]$. The key point is that when G has torsion, there will be proper G-manifolds on which the G-action is not free.

4. Solvable simply connected Lie groups

The conjecture stated in §2 above is verified for (connected) solvable simply connected Lie groups by

PROPOSITION 1. *Let G be a (connected) solvable simply connected Lie group, and let X be a G-manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*(X,G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ \downarrow & & \downarrow \\ K^*(X) & \longrightarrow & K_*[C_0(X)] \end{array}$$

in which each arrow is an isomorphism.

The proof depends on

LEMMA 2. *Let G be a (connected) solvable simply connected Lie group, and let Z be a proper G-manifold. Then there exists a G-map from Z to G.*

Proof of Lemma 2. Since the action of G on Z is proper all isotropy groups are compact. G has no non-trivial compact subgroups, so the action of G on Z is free. Therefore Z is a principal G-bundle with base Z/G. As G is itself a contractible space on which G acts freely, there is a G-map from Z to G. □

Proof of Proposition 1. In the diagram of the proposition the right vertical arrow is the Thom isomorphism of [13]. The lower horizontal arrow is the standard isomorphism which is valid for any locally compact Hausdorff topological space.