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Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
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When G has torsion, the map $K_*^T([EG \times X]/G) \to K^*(X, G)$ can fail to be an isomorphism. The simplest example of this is obtained by taking X to be a point and $G = \mathbf{Z}/2\mathbf{Z}$.

When G has torsion, $K_*^T([EG \times X]/G)$ appears to be only a first approximation to $K^*(X, G)$ and $K_*[C_0(X) \rtimes G]$. The key point is that when G has torsion, there will be proper G-manifolds on which the G-action is not free.

4. Solvable simply connected Lie groups

The conjecture stated in §2 above is verified for (connected) solvable simply connected Lie groups by

PROPOSITION 1. *Let G be a (connected) solvable simply connected Lie group, and let X be a G-manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*(X, G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ \downarrow & & \downarrow \\ K^*(X) & \longrightarrow & K_*[C_0(X)] \end{array}$$

in which each arrow is an isomorphism.

The proof depends on

LEMMA 2. *Let G be a (connected) solvable simply connected Lie group, and let Z be a proper G-manifold. Then there exists a G-map from Z to G.*

Proof of Lemma 2. Since the action of G on Z is proper all isotropy groups are compact. G has no non-trivial compact subgroups, so the action of G on Z is free. Therefore Z is a principal G-bundle with base Z/G. As G is itself a contractible space on which G acts freely, there is a G-map from Z to G. □

Proof of Proposition 1. In the diagram of the proposition the right vertical arrow is the Thom isomorphism of [13]. The lower horizontal arrow is the standard isomorphism which is valid for any locally compact Hausdorff topological space.

To define the left vertical arrow the first step is to use the lemma to construct an isomorphism

$$(1) \qquad K^*(X,G) \to K_G^*(T^*[X \times G] \oplus \pi_1^* T^* X).$$

Here G acts on $X \times G$ by

$$(x, g_1)\, g = (xg, g_1 g)$$

and $\pi_1 : X \times G \to X$ is the projection.

If (Z, ξ, f) is a K-cocycle for (X, G) then according to the lemma there exists a G-map $\psi\colon Z \to G$. Define $h\colon Z \to X \times G$ by $h(z) = (f\, z, \psi z)$ so that there is the evident commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\;h\;} & X \times G \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & X & \end{array}$$

The isomorphism (1) is

$$(Z, \xi, f) \to h_!(\xi).$$

Next, $T^*[X \times G] \oplus \pi_1^* T^* X$ has a G-invariant Spin^c-structure so by the Thom isomorphism theorem of §2, there is an isomorphism

$$(2) \qquad K_G^*(T^*[X \times G] \oplus \pi_1^* T^* X) \cong K_G^*(X \times G).$$

Finally, the action of G on $X \times G$ is free and has $[X \times G]/G = X$. This yields an isomorphism

$$(3) \qquad K_G^*(X \times G) \cong K^*(X).$$

Composing (1), (2), (3) gives the left vertical arrow of the proposition. □

REMARK 3. The two vertical arrows in the diagram of the proposition are not quite canonical. First an orientation must be chosen for the Lie algebra of G. There is no dimension shift in the horizontal arrows of the proposition. If $\epsilon = \dim(G)$, then the left vertical arrow maps $K^i(X, G)$ to $K^{i+\epsilon}(X)$, and the right vertical arrow maps $K_i[C_0(X) \rtimes G]$ to $K_{i+\epsilon}[C_0(X)]$.