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Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
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REMARK 3. For G discrete the reduced C^*-algebra of G, denoted C^*G, comes equipped with a trace. An element in C^*G is a formal sum $\sum_{g\in G} \lambda_g g$ where $\lambda_g \in \mathbf{C}$. The trace of such an element is λ_1 where 1 is the identity element of G. This trace then induces a map

$$\mathrm{tr}\colon K_0\, C^*G \to \mathbf{R}.$$

Let Z be a proper G-manifold and let D be a G-invariant elliptic operator on Z. If ξ is the symbol of D then (Z, ξ) is a K-cocycle for $(\cdot, G)$ and the Chern character defined above assigns to (Z, ξ)

$$\mathrm{ch}(Z, \xi) \in H_*(BG; \mathbf{C}).$$

Let $\epsilon\colon BG \to \cdot$ be the map of BG to a point. Identify $H_*(\cdot, \mathbf{C}) = \mathbf{C}$ and consider

$$\epsilon_*\, \mathrm{ch}(Z, \xi) \in \mathbf{C}.$$

The K-theory index of the elliptic operator D is an element of $K_0\, C^*G$

$$\mathrm{Index}(D) \in K_0\, C^*G.$$

We then have the following formula for $\mathrm{tr}[\mathrm{Index}(D)]$:

$$\mathrm{tr}[\mathrm{Index}(D)] = \epsilon_*\, \mathrm{ch}(Z, \xi).$$

For the special case when the action of G on Z is free this formula was obtained by M.F. Atiyah [3].

7. COROLLARIES OF THE ISOMORPHISM CONJECTURE

The conjecture stated in §2 above asserts that

$$\mu\colon K^*(X, G) \to K_*[C_0(X) \rtimes G]$$

is an isomorphism. Suppose that G is a discrete group and X is a point. The conjecture then asserts that $\mu\colon K^*(\cdot, G) \to K_*C^*G$ is an isomorphism where C^*G is the reduced C^*-algebra of G. Throughout this section G will be a discrete group and we shall consider some corollaries of the conjecture that $\mu\colon K^0(\cdot, G) \to K_0\, C^*G$ is an isomorphism. "*Proof*" will mean "*Proof modulo the conjecture*".

COROLLARY 1. *If G is torsion free then* $\operatorname{tr}\colon K_0 C^*G \to \mathbf{R}$ *maps* $K_0 C^*G$ *onto the integers* $\mathbf{Z}$.

"Proof". Let (Z, ξ) be a K-cocycle for $(\cdot, G)$. Let D be a G-invariant elliptic operator on Z whose symbol is ξ. By the definition of $\mu\colon K^0(\cdot, G) \to K_0 C^*G$ given in §2 above

$$\mu(Z, \xi) = \operatorname{Index}(D).$$

If G is torsion free then the action of G on Z must be free. Hence Atiyah's result applies [3] and $\operatorname{tr}[\operatorname{Index}(D)]$ must be an integer. Thus the surjectivity of $\mu\colon K^0(\cdot, G) \to K_0 C^*G$ implies that $\operatorname{tr}\colon K_0 C^*G \to \mathbf{R}$ takes on only integer values. □

COROLLARY 2. *If G is torsion free then there are no non-trivial projections in C^*G.*

"Proof". A non-trivial projections in C^*G would give an element $\alpha \in K_0 C^*G$ with $0 < \operatorname{tr}(\alpha) < 1$. □

REMARK 3. For G torsion-free abelian, Corollary 2 can be proved by applying Pontrjagin duality. At the other extreme, Pimsner and Voiculescu [27] have proved that Corollary 2 is valid for a finitely generated free group.

In the statement of Corollary 2 it is essential that C^*G be the reduced C^*-algebra of G. Corollary 2 is not valid if one uses the maximal C^*-algebra $C^*_{\max}G$.

A classical conjecture [24] in the theory of group rings is that the group ring of a torsion-free group has no (non-trivial) divisors of zero. J. Cohen has observed that Corollaries 1 and 2 may be relevant to this zero-divisor conjecture.

If G has torsion then we conjecture that $\operatorname{tr}\colon K_0 C^*G \to \mathbf{R}$ maps $K_0 C^*G$ onto the additive subgroup of $\mathbf{Q}$ generated by all rational numbers of the form $\frac{1}{n}$, where n is the order of a finite subgroup of G. This would follow from the conjectured surjectivity of $K^0(\cdot, G) \to K_0 C^*G$ plus the unproved assertion that $\operatorname{tr}[\operatorname{Index}(D)]$ can only take on such values, where D is any G-invariant elliptic operator on a proper G-manifold.

COROLLARY 4. *The Novikov conjecture on homotopy invariance of higher signatures* [11].

"Proof". Let M be a closed oriented C^∞-manifold, $G = \pi_1(M)$ and let $f\colon M \to BG$ be the classifying map of the universal covering space of M.

The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where $\mathbf{L}(M)$ is the total $\mathbf{L}$ class of TM and a is any element in $H^*(BG; \mathbf{Q})$.

Kasparov [19] and Miscenko-Fomenko [21] [22] define a map

$$K_0(BG) \to K_0\, C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0\, C^*G \\ & \searrow \quad \swarrow & \\ & K^0(\cdot, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of K-homology given in [9] is being used.) □

COROLLARY 5. *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg* [30].

For the same reason our conjecture implies the stable[1]) form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the C^*-algebra of the projective regular representation of a discrete group.

All of § 2 adapts to the projective situation where together with the G-manifold X one is given a 2-cocycle $\gamma \in Z^2(X \rtimes G, S^1)$. For simplicity we

[1]) Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture Also, Max Karoubi and I have proved that the usual (i e complex K-theory) form of Baum-Connes implies the real form of Baum-Connes