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2.3 DIRICHLET  $L$ -FUNCTIONS

For  $\chi$  a Dirichlet character with conductor  $f_\chi$ , the Dirichlet  $L$ -function for  $\chi$  is defined by

$$L(s; \chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

for  $s \in \mathbf{C}$  such that  $\Re(s) > 1$ . Note that  $L(s; \chi)$  can be continued analytically to all of  $\mathbf{C}$ , except for a pole of order 1 at  $s = 1$  when  $\chi = 1$ .

Let  $\tau(\chi)$  be a Gauss sum,

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a / f_\chi},$$

where  $i^2 = -1$ , and let

$$\delta_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

Then  $L(s; \chi)$  satisfies the functional equation

$$(7) \quad \left(\frac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\frac{s + \delta_\chi}{2}\right) L(s; \chi) = W_\chi \left(\frac{f_\chi}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s + \delta_\chi}{2}\right) L(1-s; \bar{\chi}),$$

where  $\Gamma(s)$  is the gamma function, and  $W_\chi = \frac{\tau(\chi)}{i^{\delta_\chi} \cdot \sqrt{f_\chi}}$ , having the property that  $|W_\chi| = 1$ . Since  $\Gamma(s)$  has simple poles at the negative integers,  $L(s; \chi)$  must be zero for  $s = 1 - n$ , where  $n \in \mathbf{Z}$ ,  $n \geq 1$ , such that  $n \not\equiv \delta_\chi \pmod{2}$ , except when  $\chi = 1$  and  $n = 1$ .  $L(s; \chi)$  can also be described by means of the Euler product  $L(s; \chi) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}$ , for  $s \in \mathbf{C}$  such that  $\Re(s) > 1$ . Thus  $L(s; \chi) \neq 0$  in this domain.

The generalized Bernoulli numbers,  $B_{n, \chi}$ , and the Dirichlet  $L$ -function,  $L(s; \chi)$ , share the following relationship, a proof of this being found in [13]:

**THEOREM 2.1.** *Let  $\chi$  be a Dirichlet character, and let  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Then  $L(1 - n; \chi) = -\frac{1}{n} B_{n, \chi}$ .*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.