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2.4 THE p -ADIC NUMBER FIELD

Let p be prime. We shall use \mathbf{Z}_p to represent the p -adic integers, and \mathbf{Q}_p the p -adic rationals. Let $|\cdot|_p$ denote the p -adic absolute value on \mathbf{Q}_p , normalized so that $|p|_p = p^{-1}$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of \mathbf{Q}_p . The absolute value on \mathbf{Q}_p extends uniquely to $\overline{\mathbf{Q}}_p$, however $\overline{\mathbf{Q}}_p$ is not complete with respect to the absolute value. Let \mathbf{C}_p be the completion of $\overline{\mathbf{Q}}_p$ with respect to this absolute value. Then the absolute value extends to \mathbf{C}_p , and $\overline{\mathbf{Q}}_p$ is dense in \mathbf{C}_p . We also have \mathbf{C}_p algebraically closed. Furthermore, on \mathbf{C}_p the absolute value is non-Archimedean, and so

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

for any $a, b \in \mathbf{C}_p$. Note that the two fields \mathbf{C} and \mathbf{C}_p are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of \mathbf{C}_p in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \quad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then \mathfrak{p} is a maximal ideal of \mathfrak{o} . If $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq |p|_p^s$, where $s \in \mathbf{Q}$, then $\tau \in p^s \mathfrak{o}$, and so we shall also write this as $\tau \equiv 0 \pmod{p^s \mathfrak{o}}$.

Any $n \in \mathbf{Z}$, $n > 0$, can be uniquely expressed in the form $n = \sum_{m=0}^k a_m p^m$, where $a_m \in \mathbf{Z}$, $0 \leq a_m \leq p - 1$, for $m = 0, 1, \dots, k$, and $a_k \neq 0$. For such n , we define

$$s_p(n) = \sum_{m=0}^k a_m,$$

the sum of the p -adic digits of n , and also define $s_p(0) = 0$. For any $n \in \mathbf{Z}$, let $v_p(n)$ be the highest power of p dividing n . This function is additive, and relates to the function $s_p(n)$ by means of the identity

$$(8) \quad v_p(n!) = \frac{n - s_p(n)}{p - 1},$$

which holds for all $n \geq 0$. Note that for $n \geq 1$ this implies that

$$v_p(n!) \leq \frac{n - 1}{p - 1}.$$

The definition of this function can be extended to all of \mathbf{Q} by taking $v_p(1/n) = -v_p(n)$.

Throughout we let $q = 4$ if $p = 2$, and $q = p$ otherwise. Note that there exist $\phi(q)$ distinct solutions, modulo q , to the equation $x^{\phi(q)} - 1 = 0$, and each solution must be congruent to one of the values $a \in \mathbf{Z}$, where $1 \leq a \leq q$,

$(a, p) = 1$. Thus, by Hensel's Lemma, given $a \in \mathbf{Z}$ with $(a, p) = 1$, there exists a unique $\omega(a) \in \mathbf{Z}_p$, where $\omega(a)^{\phi(q)} = 1$, such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting $\omega(a) = 0$ for $a \in \mathbf{Z}$ such that $(a, p) \neq 1$, we see that ω is actually a Dirichlet character, called the Teichmüller character, having conductor $f_\omega = q$. Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$. For $p \geq 3$, $\lim_{n \rightarrow \infty} a^{p^n} = \omega(a)$, since $a^{p^n} \equiv a \pmod{p}$ and $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$.

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character ω . If $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbf{Z}$, $a + qt \equiv a \pmod{q\mathfrak{o}}$. Thus we define

$$\omega(a + qt) = \omega(a)$$

for these values of t . We also define

$$\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$$

for such t .

Fix an embedding of the algebraic closure of \mathbf{Q} , $\overline{\mathbf{Q}}$, into \mathbf{C}_p . We may then consider the values of a Dirichlet character χ as lying in \mathbf{C}_p . For $n \in \mathbf{Z}$ we define the product $\chi_n = \chi\omega^{-n}$ in the sense of the product of characters. This implies that $f_{\chi_n} \mid f_\chi q$. However, since we can write $\chi = \chi_n\omega^n$, we also have $f_\chi \mid f_{\chi_n} q$. Thus f_χ and f_{χ_n} differ by a factor that is a power of p . In fact, either $f_{\chi_n}/f_\chi \in \mathbf{Z}$ and divides q , or $f_\chi/f_{\chi_n} \in \mathbf{Z}$ and divides q .

Let $\mathbf{Q}_p(\chi)$ denote the field generated over \mathbf{Q}_p by all values $\chi(a)$, $a \in \mathbf{Z}$. In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field $\mathbf{Q}_p(\chi)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} (B_{n+1,\chi}(p^h f_\chi) - B_{n+1,\chi}(0)).$$

From this we can obtain

LEMMA 2.3. Let $\tau \in \mathbf{C}_p$. In the field $\mathbf{Q}_p(\chi, \tau)$, for all $n \in \mathbf{Z}$, $n \geq 0$,

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n.$$

Proof. By applying Lemma 2.2 to (4), we obtain

$$B_{n, \chi} = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi(a)a^n.$$

Therefore, by (2),

$$\begin{aligned} B_{n, \chi_n}(\tau) &= \sum_{m=0}^n \binom{n}{m} \tau^{n-m} \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h \rightarrow \infty} \frac{1}{p^{hf_{\chi_n}}} \sum_{a=1}^{p^hf_{\chi_n}} \chi_n(a) \sum_{m=0}^n \binom{n}{m} \tau^{n-m} a^m. \end{aligned}$$

Since f_χ and f_{χ_n} differ by a factor that is a power of p , we must have

$$B_{n, \chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^{hf_\chi}} \sum_{a=1}^{p^hf_\chi} \chi_n(a)(a + \tau)^n,$$

and the proof is complete. \square

2.5 p -ADIC FUNCTIONS

Let K be an extension of \mathbf{Q}_p contained in \mathbf{C}_p . An infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in K$, converges in K if and only if $|a_n|_p \rightarrow 0$ as $n \rightarrow \infty$. Let $K[[x]]$ be the algebra of formal power series in x . Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in $K[[x]]$, converges at $x = \xi$, $\xi \in \mathbf{C}_p$, if and only if $|a_n \xi^n|_p \rightarrow 0$ as $n \rightarrow \infty$. Therefore whenever a power series $A(x)$ converges at some $\xi_0 \in \mathbf{C}_p$, then it must converge at all $\xi \in \mathbf{C}_p$ such that $|\xi|_p \leq |\xi_0|_p$. The following result, for double series in K , can be found in [8].