Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 46 (2000)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: p-ADIC L-FUNCTION OF TWO VARIABLES
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Kapitel: 3.1 $L_p(s,\tau;\lambda)$ FOR $\tau \in \bar{Q}_p$, $|\tau|_p \leq 1$
DOI: https://doi.org/10.5169/seals-64800

3. THE p-ADIC L-FUNCTION $L_p(s,t;\chi)$

In the following, we apply Theorem 2.7 to the sequence $\{b_n(\tau)\}_{n=0}^{\infty}$, where $b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)$, for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, to show that there exists a power series $A_\chi(s,\tau) \in K_\tau[[s]]$, $K_\tau = \mathbf{Q}_p(\chi,\tau)$, which converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. From this we can prove the existence of a p-adic function, $L_p(s,\tau;\chi)$, that interpolates the values $L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau)$ for $n \in \mathbf{Z}$, $n \geq 1$, and converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \neq 1$ if $\chi = 1$. After this we will show that there exists $L_p(s,\tau;\chi)$ for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, satisfying

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau),$$

and converging in the domain above.

3.1 $L_p(s,\tau;\chi)$ FOR $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$

Let p be prime, and let χ be a Dirichlet character with conductor f_χ. Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, and let $K_\tau = \mathbf{Q}_p(\chi,\tau)$, the field generated over $\mathbf{Q}_p$ by adjoining τ and the values $\chi(a)$, $a \in \mathbf{Z}$. Since τ and each of the $\chi(a)$ are in $\overline{\mathbf{Q}}_p$, we see that K_τ is a finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$. For each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, we shall derive our L-function $L_p(s,\tau;\chi)$ in a manner similar to that given for the derivation of $L_p(s;\chi)$ found in Chapter 3 of [13].

For $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, define the sequences $\{b_n(\tau)\}_{n=0}^{\infty}$ and $\{c_n(\tau)\}_{n=0}^{\infty}$ in K_τ according to

$$b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right),$$

and

$$c_n(\tau) = \sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}b_m(\tau).$$

In order to derive our L-function $L_p(s,\tau;\chi)$, we will prove a particular bound on the magnitude of $c_n(\tau)$, but to do so, we shall need the following:

LEMMA 3.1. *Let $m, r \in \mathbf{Z}$, with $m \geq 0$ and $r \geq 1$. Then*

$$\sum_{a=0}^{p^r-1} a^m \equiv 0 \pmod{p^{r-1}},$$

where we take $0^0 = 1$ in the case of $a = 0$ and $m = 0$.

Proof. This is obvious for $m = 0$, so assume that $m \geq 1$. We shall prove this result for the remaining values of m by induction on r.

Since any sum of elements of $\mathbf{Z}$ must also be in $\mathbf{Z}$, the lemma is true for $r = 1$. Now assume that the lemma holds for some $r \in \mathbf{Z}$, $r \geq 1$. By rewriting the sum

$$\sum_{a=0}^{p^{r+1}-1} a^m = \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} \left(u + p^r v\right)^m,$$

and reducing this modulo p^r, we obtain

$$\sum_{a=0}^{p^{r+1}-1} a^m \equiv \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} u^m \pmod{p^r}$$
$$\equiv p \sum_{u=0}^{p^r-1} u^m \pmod{p^r}.$$

By our induction hypothesis we must then have

$$\sum_{a=0}^{p^{r+1}-1} a^m \equiv 0 \pmod{p^r},$$

and the lemma follows. □

LEMMA 3.2. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 0$. For all $h \in \mathbf{Z}$, $h \geq 1$,*

$$\frac{1}{q^h f_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) \left(\langle a + q\tau \rangle - 1\right)^n \equiv 0 \pmod{f_\chi^{-1} p^{-1} q^{n-1} \mathfrak{o}}.$$

Proof. This is obvious for $n = 0$ since writing

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \sum_{a=1}^{q^h f_\chi} \chi(a) - \sum_{a=1}^{p^{-1} q^h f_\chi} \chi(pa)$$

allows us to derive

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{q^h f_\chi} \chi(a) = \begin{cases} p^{-1} q^h (p-1), & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

So let us assume that $n \geq 1$.

Let $h = 1$. Then $\langle a+q\tau\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for all $a \in \mathbf{Z}$ such that $(a,p) = 1$ implies that

$$\left(\langle a + q\tau\rangle - 1\right)^n \equiv 0 \pmod{q^n\mathfrak{o}},$$

and the lemma holds for this case.

Now assume that $h \geq 1$. We can rewrite our sum as follows:

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)(\langle a+q\tau\rangle-1)^n = \sum_{v=0}^{q^{h-1}-1} \sum_{\substack{u=1\\(u+vqf_\chi,p)=1}}^{qf_\chi} \chi(u+vqf_\chi)(\langle u+vqf_\chi+q\tau\rangle-1)^n .$$

Since $|\tau|_p \leq 1$, we can write

$$\begin{aligned}\langle u + vqf_\chi + q\tau\rangle &= \left(u + vqf_\chi + q\tau\right)\omega^{-1}\left(u + vqf_\chi + q\tau\right)\\ &= (u + q\tau)\,\omega^{-1}(u + q\tau) + vqf_\chi\omega^{-1}(u + q\tau)\\ &= \langle u + q\tau\rangle + vqf_\chi\omega^{-1}(u).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\left(\langle a + q\tau\rangle - 1\right)^n\\ &\qquad = \sum_{\substack{u=1\\(u,p)=1}}^{qf_\chi} \chi(u) \sum_{v=0}^{q^{h-1}-1} \left(\langle u + q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u)\right)^n .\end{aligned}$$

By expanding, the inner sum on the right can be written

$$\begin{aligned}&\sum_{v=0}^{q^{h-1}-1} (\langle u + q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u))^n\\ &\qquad = \sum_{k=0}^{n} \binom{n}{k} (\langle u + q\tau\rangle - 1)^{n-k} q^k f_\chi^k \omega^{-k}(u) \sum_{v=0}^{q^{h-1}-1} v^k .\end{aligned}$$

Since $(u,p) = 1$, we obtain the equivalence

$$q^k\left(\langle u + q\tau\rangle - 1\right)^{n-k} \equiv 0 \pmod{q^n\mathfrak{o}}$$

for each k, $0 \leq k \leq n$. Furthermore, by Lemma 3.1

$$\sum_{v=0}^{q^{h-1}-1} v^k \equiv 0 \pmod{p^{-1}q^{h-1}}$$

for each such k. Therefore

$$\sum_{v=0}^{q^{h-1}-1} \left(\langle u+q\tau\rangle - 1 + vqf_{\chi}\omega^{-1}(u)\right)^{n} \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}}.$$

This implies that

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^{h}f_{\chi}} \chi(a)(\langle a+q\tau\rangle - 1)^{n} \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}},$$

yielding the result. □

We now derive our bound on the magnitude of $c_n(\tau)$.

PROPOSITION 3.3. *For all* $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, *and for* $n \in \mathbf{Z}$, $n \ge 0$, *we have* $|c_n(\tau)|_p \le |pqf_{\chi}|_p^{-1}|q|_p^{n}$.

Proof. This follows in a manner similar to that given for the proof of the bound $|c_n(0)|_p \le \left|q^2 f_{\chi}\right|_p^{-1}|q|_p^{n}$ found in [13] (Lemma 4 of Chapter 3). However, in this case we use Lemma 2.3 and the properties of χ and ω to derive

$$b_n(\tau) = \lim_{h\to\infty} \frac{1}{q^{h}f_{\chi}} \sum_{\substack{a=1\\(a,p)=1}}^{q^{h}f_{\chi}} \chi(a)\langle a+q\tau\rangle^{n}$$

for each $n \ge 0$, and thus

$$c_n(\tau) = \lim_{h\to\infty} \frac{1}{q^{h}f_{\chi}} \sum_{\substack{a=1\\(a,p)=1}}^{q^{h}f_{\chi}} \chi(a)\left(\langle a+q\tau\rangle - 1\right)^{n}$$

for each such n. From Lemma 3.2 we obtain

$$c_n(\tau) \equiv 0 \pmod{f_{\chi}^{-1}p^{-1}q^{n-1}\mathfrak{o}},$$

and thus the result. □

For our immediate concern we only need this proposition to hold for all $\tau \in \overline{\mathbf{Q}}_p$ such that $|\tau|_p \le 1$. However, later on we shall need it in the form in which we have it.

We are now ready to begin the construction of our L-function.

THEOREM 3.4. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a power series* $A_\chi(s,\tau)$ *in* $K_\tau[[s]]$ *such that the power series converges on* $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, *and for each* $n \in \mathbf{Z}$, $n \geq 0$, $A_\chi(n,\tau)$ *satisfies*

$$A_\chi(n,\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right).$$

Proof. By Proposition 3.3, $|c_n(\tau)|_p \leq C|q|_p^n$ for all $n \geq 0$, where $C = |pqf_\chi|_p^{-1}$. Therefore we can apply Theorem 2.7 to the sequences $\{b_n(\tau)\}_{n=0}^\infty$ and $\{c_n(\tau)\}_{n=0}^\infty$ in $K_\tau = \mathbf{Q}_p(\chi,\tau)$, and for $\rho = |q|_p < |p|_p^{1/(p-1)}$, yielding this result. □

Let us denote $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$.

THEOREM 3.5. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a unique* p*-adic, meromorphic function* $L_p(s,\tau;\chi)$ *that can be expressed in the form*

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *having coefficients* $a_n(\tau) \in \mathbf{Q}_p(\chi,\tau)$, *with*

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

Furthermore, for each $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right).$$

Proof. Let

(13) $$L_p(s,\tau;\chi) = \frac{1}{s-1}A_\chi(1-s,\tau)$$

with the $A_\chi(s,\tau)$ as in Theorem 3.4. Then from the properties of $A_\chi(s,\tau)$, the power series must converge in the given domain, and for $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}A_\chi(n,\tau) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right).$$

Note that

$$\begin{aligned} a_{-1}(\tau) &= A_\chi(0,\tau) = B_{0,\chi}(q\tau) - \chi(p)p^{-1}B_{0,\chi}\left(p^{-1}q\tau\right) \\ &= (1-\chi(p)p^{-1})B_{0,\chi}, \end{aligned}$$

and thus

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

The uniqueness of $L_p(s, \tau; \chi)$ follows from Lemma 2.5. □

At this point we have not completed our goal of showing that the p-adic function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. In order to prove this, we will need to study the coefficients, $a_n(\tau)$, of the power series expansion of $L_p(s, \tau; \chi)$ for each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. From the results of this we will show that the function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for any sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to τ, the values $L_p(1-n, \tau_i; \chi)$ converge to $L_p(1-n, \tau; \chi)$ for each $n \in \mathbf{Z}$, $n \geq 1$.

3.2 $L_p(s, \tau; \chi)$ FOR $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$

Our previous work has been for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. To extend this result to all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we need to find a way to express $a_n(\tau)$ so that it can be defined for these values of τ.

For $k \in \mathbf{Z}$, $k \geq 0$, the Stirling numbers of the first kind, $s(n, k)$, are defined by the generating function

$$\sum_{n=0}^{\infty} s(n, k) \frac{t^n}{n!} = \frac{1}{k!} (\log(1+t))^k . \tag{14}$$

Since the power series expansion of $\log(1+t)$ lacks a constant term, we must have $s(n, k) = 0$ whenever $0 \leq n < k$. We also have $s(n, n) = 1$ for all $n \geq 0$. The $s(n, k)$ are integers, where $n, k \in \mathbf{Z}$, $n \geq 0$, $k \geq 0$, and they satisfy the relation

$$\binom{x}{n} = \frac{1}{n!} \sum_{k=0}^{n} s(n, k) x^k . \tag{15}$$

For additional information on Stirling numbers of the first kind we refer the reader to [6], pp. 214–217.

LEMMA 3.6. *Let* $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. *For* $n \in \mathbf{Z}$, $n \geq -1$,

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau) .$$

Proof. From Corollary 2.8 we can write