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*where the power series converges in the domain* $\mathfrak{D}$, *and*

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases} \quad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a $p$-adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

## 4. Properties of $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take $p$ prime and $\chi$ a Dirichlet character with conductor $f_\chi$.

### 4.1 A symmetry property in $t$

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t) , \tag{18}$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right) , \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t) . \tag{20}$$

LEMMA 4.1. *For all* $n \in \mathbf{Z}$, $n \geq 0$, *we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1} .$$

*Proof.* This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1} .$$

Any $m$ such that $n-m$ is even must have the same parity as $n$. Thus

$$B_{n,1}(-t) = (-1)^n \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + (-1)^{n-1} n B_{1,1} t^{n-1}$$
$$= (-1)^n B_{n,1}(t) - 2(-1)^n n B_{1,1} t^{n-1}.$$

From the value $B_{1,1} = -B_1 = 1/2$, the lemma then follows. □

LEMMA 4.2. *For all* $n \in \mathbf{Z}$, $n \geq 0$,

$$b_n(-t) = \chi(-1) b_n(t).$$

*Proof.* This is obviously true for $n=0$ since

$$b_0(t) = \left(1 - \chi(p)p^{-1}\right) B_{0,\chi},$$

and $B_{0,\chi} = 0$ except when $\chi = 1$, in which case $B_{0,1} = 1$. So we can assume that $n \geq 1$.

First consider the case of $\chi_n = 1$. This implies that $\chi = \omega^n$. By Lemma 4.1,

$$\begin{aligned} b_n(-t) &= B_{n,1}(-qt) - p^{n-1} B_{n,1}\left(-p^{-1}qt\right) \\ &= (-1)^n B_{n,1}(qt) - (-1)^n n (qt)^{n-1} \\ &\quad - p^{n-1}\left((-1)^n B_{n,1}\left(p^{-1}qt\right) - (-1)^n n \left(p^{-1}qt\right)^{n-1}\right) \\ &= (-1)^n \left(B_{n,1}(qt) - p^{n-1} B_{n,1}\left(p^{-1}qt\right)\right) \\ &= (-1)^n b_n(t). \end{aligned}$$

Since $\chi = \omega^n$ and $\omega(-1) = -1$, the lemma holds for $\chi_n = 1$.

Now suppose that $\chi_n \neq 1$. Then, from (3),

$$\begin{aligned} b_n(-t) &= B_{n,\chi_n}(-qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(-p^{-1}qt\right) \\ &= (-1)^n \chi_n(-1)\left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= (-1)^n \chi_n(-1) b_n(t). \end{aligned}$$

Note that $\chi_n = \chi\omega^{-n}$, which implies that $\chi_n(-1) = (-1)^n \chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$.

Since the lemma holds for both $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. □

Using this result, we can prove

THEOREM 4.3. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi).$$

*Proof.* From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1)b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1)c_n(t).$$

From (16), whenever $n \geq -1$,

$$a_n(-t) = \chi(-1)a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi). \qquad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

## 4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in $t$ will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$. *Then for* $n \in \mathbf{Z}$, $n \geq 1$,

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = -\frac{1}{n}\left(1 - \chi(p)p^{-1}\right) B_{0,\chi}.$$

*Proof.* Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s - 1} + \sum_{m=0}^{\infty} a_m(t)(s - 1)^m,$$

where $a_{-1}(t) = (1 - \chi(p)p^{-1})B_{0,\chi}$. Thus

$$\lim_{s\to 1}(s-1)L_p(s,t;\chi) = \left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

Now let $n \in \mathbf{Z}$, $n \geq 1$, and consider

$$\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) = \lim_{s\to 1}\binom{n-s}{n}L_p(s,t;\chi)\,.$$

If $n=1$, then we write this as

$$\lim_{s\to 1}(1-s)L_p(s,t;\chi) = -\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

If $n \geq 2$, then

$$\frac{1}{n!}\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i) = \frac{1}{n},$$

which implies that

$$\begin{aligned}\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) &= \frac{1}{n!}\left(\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i)\right)\left(\lim_{s\to 1}(1-s)L_p(s,t;\chi)\right)\\ &= -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.\end{aligned}$$

Therefore the lemma holds for all $n \geq 1$. □

Now, because $L_p(s,t;1)$ is undefined when $s=1$, the quantity

$$\binom{-s}{n}L_p(s+n,t;1)$$

is undefined when $s = 1-n$, for $n \in \mathbf{Z}$, $n \geq 1$. However, Lemma 4.4 shows that this quantity exists as $s \to 1-n$. In the following we will encounter expressions that involve $\binom{-s}{n}L_p(s+n,t;\chi)$, and because of Lemma 4.4 we shall assume the understanding that

$$\left.\binom{-s}{n}L_p(s+n,t;\chi)\right|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}$$

for $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.5. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m)\,. \tag{21}$$

*Proof.* Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $k \in \mathbf{Z}$, $k \geq 1$. Then

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-k+m;\chi_m) = -\frac{1}{k}q^k t^k(1-\chi_k(p)p^{-1})B_{0,\chi_k} + \sum_{m=0}^{k-1}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m).$$

By evaluating the $L$-function, we obtain

$$\binom{k-1}{m}L_p(1-(k-m);\chi_m) = -\frac{1}{k}\binom{k}{m}\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

and thus

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p\big(1-(k-m);\chi_m\big) = -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}q^m t^m\big(1-\chi_k(p)p^{k-m-1}\big)B_{k-m,\chi_k},$$

which implies that the sum converges for $s=1-k$. Breaking this into two sums

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m) &= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}q^m t^m + \frac{1}{k}\chi_k(p)p^{k-1}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}p^{-m}q^m t^m \\ &= -\frac{1}{k}\left(B_{k,\chi_k}(qt) - \chi_k(p)p^{k-1}B_{k,\chi_k}\left(p^{-1}qt\right)\right) \\ &= L_p(1-k,t;\chi)\ .\end{aligned}$$

Thus (21) holds for a sequence $\{1-k\}_{k=1}^{\infty}$ that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all $s$ in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in $s$, of each of the functions on either side of (21) contain $\mathfrak{D}$, except $s\neq 1$ when $\chi=1$.

This is obvious for the function $L_p(s,t;\chi)$. Consider the function

$$\sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m) = \sum_{m=0}^{\infty}\sum_{n=-1}^{\infty}\binom{-s}{m}q^m t^m a_{n,\chi_m}(s+m-1)^n.$$

We have seen that this sum converges for $s=1-k$, where $k\in\mathbf{Z}$, $k\geq 1$. Now we need to show that it converges for $s=\xi$, where $\xi\in\mathfrak{D}$, $\xi\neq 1$ if $\chi=1$, and $\xi\neq 1-k$ for $k\in\mathbf{Z}$, $k\geq 1$. So let $\xi$ satisfy these restrictions, and let $\epsilon > 0$. Note that $|\xi - 1|_p < r$, where $r = |p|_p^{1/(p-1)}|q|_p^{-1}$. Let $r_0 \in \mathbf{R}$, $0 \le r_0 < r$, such that $|\xi - 1|_p = r_0$. Then for any $m \in \mathbf{Z}$, $m \ge 0$,

$$\begin{aligned} |\xi + m - 1|_p &\le \max\left\{|m|_p, |\xi - 1|_p\right\} \\ &\le \max\{1, r_0\}, \end{aligned}$$

implying that $\xi + m \in \mathfrak{D}$, $\xi + m \ne 1$. Let $\delta \in \mathbf{R}$ such that $r^\delta = \max\{1, r_0\}$. Then $0 \le \delta < 1$, and

$$|\xi + m - 1|_p \le r^\delta. \tag{22}$$

Let $N_1 \in \mathbf{Z}$ such that

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any $m \in \mathbf{Z}$, $m \ge 1$, such that $m \ge N_1$, we must also have

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For $m \in \mathbf{Z}$, $m \ge 1$, consider

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p.$$

Note that, by (22),

$$\begin{aligned} \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^{m} \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\le |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \ge |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if $m \ge N_1$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p < \epsilon.$$

Now let $N_2 \in \mathbf{Z}$ such that

$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)N_2/(p-1)} |q|_p^{(1-\delta)N_2} < \epsilon .$$

Then we must also have

$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} < \epsilon$$

for any $m, n \in \mathbf{Z}$ such that $m \geq 0$, $n \geq 0$, and $\max\{m, n\} \geq N_2$. Let us consider

$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq \left| \binom{-\xi}{m} \right|_p |q|_p^m |a_{n,\chi_m}|_p |\xi + m - 1|_p^n ,$$

where $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq 0$. For all $m \geq 0$,

$$\left| \binom{-\xi}{m} \right|_p \leq |m!|_p^{-1} r^{\delta m},$$

and by utilizing this along with (17) and (22), our expression becomes

$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq |m!(n+1)!|_p^{-1} |f_\chi p|_p^{-1} r^{\delta(m+n)} |q|_p^{m+n} .$$

Since

$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$

we obtain

$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq |f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} .$$

Thus if $\max\{m, n\} \geq N_2$, then

$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p < \epsilon .$$

Let $N = \max\{N_1, N_2\}$, and let $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq -1$. Then for $\max\{m, n\} \geq N$, it must be true that

$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p < \epsilon .$$

Thus, by Proposition 2.4, the sum

$$\sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n$$

must converge. This implies that the function on the right of (21) must converge for all $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and the theorem must then hold. □

Since we can now express $L_p(s, t; \chi)$ in terms of a power series in $t$, we can take a derivative of this function with respect to $t$.

LEMMA 4.6. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n),$$

*for* $n \in \mathbf{Z}$, $n \ge 0$.

*Proof.* If $n = 0$, then the lemma is obviously true. So consider $n = 1$. Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s, t; \chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s+m; \chi_m).$$

Now,

$$m \binom{-s}{m} = -s \binom{-s-1}{m-1},$$

so that

$$\begin{aligned} \frac{\partial}{\partial t} L_p(s, t; \chi) &= \sum_{m=1}^{\infty} (-s) \binom{-s-1}{m-1} q^m t^{m-1} L_p(s+m; \chi_m) \\ &= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p\left(s+1+m; \chi_{1+m}\right) \\ &= -qs L_p(s+1, t; \chi_1). \end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n)$$

for some $n \in \mathbf{Z}$, $n \ge 1$. Then

$$\begin{aligned} \frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) &= \frac{\partial}{\partial t}\left(\frac{\partial^n}{\partial t^n} L_p(s, t; \chi)\right) \\ &= n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n). \end{aligned}$$

From the case for $n = 1$, we see that

$$\begin{aligned} n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n) &= n! q^n \binom{-s}{n} (-s-n) q L_p\left(s+n+1, t; \chi_{n+1}\right) \\ &= (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right). \end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) = (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right),$$

and the lemma must hold by induction. □

With this result, we can derive a more general power series expansion of $L_p(s,t;\chi)$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m) .$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

*Proof.* It follows from the Taylor series expansion of $L_p(s,t;\chi)$ in the variable $t$ about $\alpha$ (see Proposition 2.6) that we can write $L_p(s,t;\chi)$ in the form

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \beta_m (t-\alpha)^m,$$

where

$$\beta_m = \frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi)\bigg|_{t=\alpha} .$$

From Lemma 4.6

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi) = \binom{-s}{m} q^m L_p(s+m,t;\chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s+m,\alpha;\chi_m),$$

completing the proof. □

## 4.3 RELATING $L_p(s,t;\chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s,t;\chi)$.

For the character $\chi$, let $F_0 = \operatorname{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let $F$ be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, t+F; \chi) - L_p(s, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} . \tag{23}$$

*Proof.* Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then from (18),

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\frac{1}{n}\left(b_n(t+F) - b_n(t)\right) .$$

Now, (19) implies

$$\begin{aligned} b_n(t+F) - b_n(t) &= \left(B_{n,\chi_n}(q(t+F)) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q(t+F)\right)\right) \\ &\quad - \left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= \left(B_{n,\chi_n}(q(t+F)) - B_{n,\chi_n}(qt)\right) \\ &\quad - \chi_n(p)p^{n-1}\left(B_{n,\chi_n}\left(p^{-1}q(t+F)\right) - B_{n,\chi_n}\left(p^{-1}qt\right)\right) . \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} b_n(t+F)&-b_n(t) \\ &= n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\chi_n(p)p^{n-1}\sum_{a=1}^{p^{-1}qF} \chi_n(a)(a+p^{-1}qt)^{n-1} \\ &= n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\sum_{\substack{a=1 \\ p \mid a}}^{qF} \chi_n(a)(a+qt)^{n-1} . \end{aligned}$$

Therefore,

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a+qt)^{n-1} .$$

Now, $\chi_n = \chi_1\omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a+qt)^{n-1} &= \chi_1(a)\omega^{-(n-1)}(a)(a+qt)^{n-1} \\ &= \chi_1(a)\langle a+qt \rangle^{n-1} . \end{aligned}$$

Thus

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt \rangle^{n-1},$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all $s$ in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable $s$, of the functions on the left of (23) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$. Consider now the function

$$-\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-s} = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-1}\langle a+qt\rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a+qt\rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a,p) = 1$, we need only show that each such function is analytic on $\mathfrak{D}$, and the proof will be complete.

The quantity $\langle a+qt\rangle^{1-s}$ can be written as

$$\langle a+qt\rangle^{1-s} = \exp\left((1-s)\log\langle a+qt\rangle\right),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a+qt\rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m.$$

Since $\langle a+qt\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for $a \in \mathbf{Z}$, $(a,p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log\langle a+qt\rangle \equiv 0 \pmod{q\mathfrak{o}}$ for such $a$ and $t$. Thus

$$\left|\frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m\right|_p \leq \left|\frac{1}{m!}q^m(s-1)^m\right|_p$$

for all $m$. By (8) we can write

$$\begin{aligned}\left|\frac{1}{m!}q^m(s-1)^m\right|_p &\leq \left|p^{-m/(p-1)}q^m(s-1)^m\right|_p \\ &= \left|p^{-1/(p-1)}q(s-1)\right|_p^m.\end{aligned}$$

Thus if

$$\left|p^{-1/(p-1)}q(s-1)\right|_p < 1,$$

then

$$\left|\frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m\right|_p \to 0$$

as $m \to \infty$. So whenever $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain $\mathfrak{D}$, except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. □

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s}.$$

*Proof.* This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character $\chi$. □

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let $\Delta_c$ denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \operatorname{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right).$$

This is the polynomial structure that we utilized with respect to generalizing the $p$-adic $L$-functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either $\chi$ or $p$.

THEOREM 4.10. *Let $n$, $c$, and $k$ be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of $n$.*

*Proof.* Since $\Delta_c$ is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\Delta_c^k\langle a\rangle^{n-1},$$

where $F$ is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k\beta_{n,\chi}(F) - \Delta_c^k\beta_{n,\chi}(0) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\Delta_c^k\langle a\rangle^{n}.$$

Note that

$$\Delta_c^k \langle a\rangle^n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \langle a\rangle^{n+mc} = \langle a\rangle^n \left(\langle a\rangle^c - 1\right)^k . \tag{24}$$

Now, $\langle a\rangle \equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a\rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k \langle a\rangle^n \equiv 0 \pmod{q^k\mathbf{Z}_p} .$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k\mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Also, since $\langle a\rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$,

$$q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1} \left(\frac{\langle a\rangle^c - 1}{q}\right)^k \tag{25}$$

implies that the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of $n$.

Let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each $i$, such that $\tau_i \to \tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i) \to \beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i\to\infty} \left(\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0)\right) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) .$$

The left side of this equality is 0 modulo $q^k\mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k\mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Furthermore, for $n'$ a positive integer,

$$\begin{aligned}\lim_{i\to\infty} &\left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)\right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n',\chi}(0)\right)\right) \\ &= \left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)\right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n',\chi}(0)\right)\right) .\end{aligned}$$

Since $\tau_i \in pq^{-1}F_0\mathbf{Z}$ for each $i$, the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of $n$. □

THEOREM 4.11. *Let $n$, $c$, $k$, and $k'$ be positive integers with $k \equiv k'$ $(\mathrm{mod}\ p-1)$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$. Then*

$$q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

*Proof.* Let $k$ and $k'$ be positive integers such that $k \equiv k'$ $(\mathrm{mod}\ p-1)$. Without loss of generality, we can assume that $k \ge k'$. From (25),

$$\begin{aligned}
&\left(q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right)\\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^k + \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^{k'}\\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^{k'}\left(\left(\frac{\langle a\rangle^c-1}{q}\right)^{k-k'}-1\right),
\end{aligned}$$

where $F$ is a positive multiple of $pq^{-1}F_0$. If $a$ is such that

$$\langle a\rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a\rangle^c-1}{q}\right)^{k-k'} - 1 \equiv 0 \pmod{p\mathbf{Z}_p},$$

since $k-k' \equiv 0$ $(\mathrm{mod}\ p-1)$. Thus

$$q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each $i$, such that $\tau_i \to \tau$. Consider

$$\begin{aligned}
&\lim_{i\to\infty}\left((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))\right)\\
&\quad = \left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right).
\end{aligned}$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

THEOREM 4.12. *Let $n$, $c$, and $k$ be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

*and, modulo $q\mathbf{Z}_p[\chi]$, is independent of $n$.*

*Proof.* We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k}L_p(1-n,F;\chi) = \binom{q^{-1}\Delta_c}{k}L_p(1-n;\chi) - \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n-1},$$

where $F$ is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n}.$$

Utilizing (15), we can write

$$\begin{aligned}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\Delta_c^m\langle a\rangle^n \\ &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\langle a\rangle^n\left(\langle a\rangle^c - 1\right)^m,\end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n = \langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k}.$$

Since $q^{-1}(\langle a\rangle^c - 1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a,p) = 1$, we see that

$$\langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a\rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of $n$.

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each $i$, such that $\tau_i \to \tau$. We are working with polynomials, so that

$$\lim_{i\to\infty}\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) = \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0),$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let $n'$ be a positive integer, and consider

$$\lim_{i\to\infty}\left(\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\binom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right)$$
$$= \left(\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\binom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right).$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of $n$. □

## 4.4 Generalized Bernoulli power series

In [9] we find a definition of ordinary Bernoulli numbers of negative index, $B_{-n}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n}, \tag{26}$$

where the limit is taken in a $p$-adic sense. Note that $\phi(p^k) \to 0$ in $\mathbf{Z}_p$ as $k \to \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k\to\infty}\left(1 - p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n} \\ &= \lim_{k\to\infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \omega^{-n}\right) \\ &= nL_p\left(n+1; \omega^{-n}\right). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. *Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then*

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

*where each prime $r$ is taken to be a rational prime.*

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) \mid n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

*Proof.* By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) \mid \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) \mid n$ if and only if $(p-1) \mid (\phi(p^k) - n)$. Therefore, if $k$ is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

and the result follows from (26). □

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{C}_p$ according to

$$B_{-n,\chi} = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi} , \tag{27}$$

where the limit is once again taken in a $p$-adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m,\chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters $\chi$ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n,\chi} &= \lim_{k \to \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n,\chi_{\phi(p^k)}} \\ &= \lim_{k \to \infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \chi_n\right) \\ &= n L_p(n+1; \chi_n) , \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n,1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and $k$ sufficiently large, we obtain $B_{-n,1} = B_{-n}$ for all such $n$.

If $k \geq 2$, then $\phi(p^k)$ is even. Thus $n$ and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even}. \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let $\chi$ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

*Proof.* Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k \to \infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the $p$-adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the $p$-adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1} B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)\right) \\ &= \lim_{k\to\infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= n L_p(n+1, t; \chi_n). \end{aligned}$$

Since $L_p(n+1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such $t$. Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p(n+m+1; \chi_{n+m}) \\ &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n+m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \leq \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m, \tag{28}$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \geq 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

*Proof.* By definition, since $|mf_\chi|_p \leq |q|_p$,

$$\begin{aligned} B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi}\left(mf_\chi\right) - B_{\phi(p^k)-n,\chi}(0)\right) \\ &= \lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k-1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a,p) = 1$. These imply that

$$\lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. $\square$

THEOREM 4.16. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all $\chi$ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,*

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t)\,.$$

*Proof.* Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m ,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod 2$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n + \delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t)\,, \end{aligned}$$

and the proof is complete. □

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