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Autor: Fox, Glenn J.
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where the power series converges in the domain $\mathfrak{D}$, and

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases} \qquad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a p-adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

4. Properties of $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ.

4.1 A symmetry property in t

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t) , \tag{18}$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right) , \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t) . \tag{20}$$

Lemma 4.1. *For all $n \in \mathbf{Z}$, $n \geq 0$, we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1} .$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1} .$$

Any m such that $n-m$ is even must have the same parity as n. Thus

$$\begin{aligned} B_{n,1}(-t) &= (-1)^n \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + (-1)^{n-1} n B_{1,1} t^{n-1} \\ &= (-1)^n B_{n,1}(t) - 2(-1)^n n B_{1,1} t^{n-1} . \end{aligned}$$

From the value $B_{1,1} = -B_1 = 1/2$, the lemma then follows. □

LEMMA 4.2. *For all* $n \in \mathbf{Z}$, $n \geq 0$,

$$b_n(-t) = \chi(-1) b_n(t).$$

Proof. This is obviously true for $n = 0$ since

$$b_0(t) = \left(1 - \chi(p) p^{-1}\right) B_{0,\chi} ,$$

and $B_{0,\chi} = 0$ except when $\chi = 1$, in which case $B_{0,1} = 1$. So we can assume that $n \geq 1$.

First consider the case of $\chi_n = 1$. This implies that $\chi = \omega^n$. By Lemma 4.1,

$$\begin{aligned} b_n(-t) &= B_{n,1}(-qt) - p^{n-1} B_{n,1}\left(-p^{-1} qt\right) \\ &= (-1)^n B_{n,1}(qt) - (-1)^n n (qt)^{n-1} \\ &\quad - p^{n-1}\left((-1)^n B_{n,1}\left(p^{-1} qt\right) - (-1)^n n \left(p^{-1} qt\right)^{n-1}\right) \\ &= (-1)^n \left(B_{n,1}(qt) - p^{n-1} B_{n,1}\left(p^{-1} qt\right)\right) \\ &= (-1)^n b_n(t) . \end{aligned}$$

Since $\chi = \omega^n$ and $\omega(-1) = -1$, the lemma holds for $\chi_n = 1$.

Now suppose that $\chi_n \neq 1$. Then, from (3),

$$\begin{aligned} b_n(-t) &= B_{n,\chi_n}(-qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(-p^{-1} qt\right) \\ &= (-1)^n \chi_n(-1) \left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right)\right) \\ &= (-1)^n \chi_n(-1) b_n(t) . \end{aligned}$$

Note that $\chi_n = \chi\omega^{-n}$, which implies that $\chi_n(-1) = (-1)^n \chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$.

Since the lemma holds for both $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. □

Using this result, we can prove

THEOREM 4.3. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi).$$

Proof. From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1) b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1) c_n(t).$$

From (16), whenever $n \geq -1$,

$$a_n(-t) = \chi(-1) a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi). \quad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in t will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$. *Then for* $n \in \mathbf{Z}$, $n \geq 1$,

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = -\frac{1}{n} \left(1 - \chi(p) p^{-1}\right) B_{0,\chi}.$$

Proof. Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s - 1} + \sum_{m=0}^{\infty} a_m(t)(s - 1)^m,$$

where $a_{-1}(t) = (1 - \chi(p) p^{-1}) B_{0,\chi}$. Thus