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Kapitel: 4.2 $L_p(s,t;\lambda)$ AS A POWER SERIES IN $t – \alpha, \quad \alpha \in C_p, \quad $|\alpha|_p \leq 1$

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THEOREM 4.3. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi).$$

Proof. From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1) b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1) c_n(t).$$

From (16), whenever $n \ge -1$,

$$a_n(-t) = \chi(-1) a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi). \qquad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \le 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in t will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$. *Then for* $n \in \mathbf{Z}$, $n \ge 1$,

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = -\frac{1}{n}\left(1 - \chi(p) p^{-1}\right) B_{0,\chi}.$$

Proof. Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s - 1} + \sum_{m=0}^{\infty} a_m(t)(s - 1)^m,$$

where $a_{-1}(t) = (1 - \chi(p) p^{-1}) B_{0,\chi}$. Thus

$$\lim_{s\to 1}(s-1)L_p(s,t;\chi) = \left(1-\chi(p)p^{-1}\right)B_{0,\chi}\,.$$

Now let $n \in \mathbf{Z}$, $n \geq 1$, and consider

$$\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) = \lim_{s\to 1}\binom{n-s}{n}L_p(s,t;\chi)\,.$$

If $n=1$, then we write this as

$$\lim_{s\to 1}(1-s)L_p(s,t;\chi) = -\left(1-\chi(p)p^{-1}\right)B_{0,\chi}\,.$$

If $n \geq 2$, then

$$\frac{1}{n!}\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i) = \frac{1}{n},$$

which implies that

$$\begin{aligned}\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) &= \frac{1}{n!}\left(\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i)\right)\left(\lim_{s\to 1}(1-s)L_p(s,t;\chi)\right)\\ &= -\frac{1}{n}\left(1-\chi(p)p^{-1}\right)B_{0,\chi}\,.\end{aligned}$$

Therefore the lemma holds for all $n \geq 1$. $\square$

Now, because $L_p(s,t;1)$ is undefined when $s=1$, the quantity

$$\binom{-s}{n}L_p(s+n,t;1)$$

is undefined when $s = 1-n$, for $n \in \mathbf{Z}$, $n \geq 1$. However, Lemma 4.4 shows that this quantity exists as $s \to 1-n$. In the following we will encounter expressions that involve $\binom{-s}{n}L_p(s+n,t;\chi)$, and because of Lemma 4.4 we shall assume the understanding that

$$\left.\binom{-s}{n}L_p(s+n,t;\chi)\right|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right)B_{0,\chi}$$

for $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.5. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m)\,. \tag{21}$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $k \in \mathbf{Z}$, $k \geq 1$. Then

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-k+m;\chi_m) = -\frac{1}{k}q^k t^k(1-\chi_k(p)p^{-1})B_{0,\chi_k}$$
$$+\sum_{m=0}^{k-1}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m).$$

By evaluating the L-function, we obtain

$$\binom{k-1}{m}L_p(1-(k-m);\chi_m) = -\frac{1}{k}\binom{k}{m}\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

and thus

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p\big(1-(k-m);\chi_m\big)$$
$$= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}q^m t^m\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

which implies that the sum converges for $s=1-k$. Breaking this into two sums

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m)$$
$$= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}q^m t^m + \frac{1}{k}\chi_k(p)p^{k-1}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}p^{-m}q^m t^m$$
$$= -\frac{1}{k}\left(B_{k,\chi_k}(qt) - \chi_k(p)p^{k-1}B_{k,\chi_k}\left(p^{-1}qt\right)\right)$$
$$= L_p(1-k,t;\chi).$$

Thus (21) holds for a sequence $\{1-k\}_{k=1}^{\infty}$ that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in s, of each of the functions on either side of (21) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$.

This is obvious for the function $L_p(s,t;\chi)$. Consider the function

$$\sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m) = \sum_{m=0}^{\infty}\sum_{n=-1}^{\infty}\binom{-s}{m}q^m t^m a_{n,\chi_m}(s+m-1)^n.$$

We have seen that this sum converges for $s=1-k$, where $k\in\mathbf{Z}$, $k\geq 1$. Now we need to show that it converges for $s=\xi$, where $\xi\in\mathfrak{D}$, $\xi\neq 1$ if $\chi=1$, and $\xi\neq 1-k$ for $k\in\mathbf{Z}$, $k\geq 1$. So let ξ satisfy these restrictions, and let $\epsilon > 0$. Note that $|\xi - 1|_p < r$, where $r = |p|_p^{1/(p-1)}|q|_p^{-1}$. Let $r_0 \in \mathbf{R}$, $0 \leq r_0 < r$, such that $|\xi - 1|_p = r_0$. Then for any $m \in \mathbf{Z}$, $m \geq 0$,

$$\begin{aligned} |\xi + m - 1|_p &\leq \max\left\{|m|_p, |\xi - 1|_p\right\} \\ &\leq \max\{1, r_0\}, \end{aligned}$$

implying that $\xi + m \in \mathfrak{D}$, $\xi + m \neq 1$. Let $\delta \in \mathbf{R}$ such that $r^\delta = \max\{1, r_0\}$. Then $0 \leq \delta < 1$, and

$$|\xi + m - 1|_p \leq r^\delta. \tag{22}$$

Let $N_1 \in \mathbf{Z}$ such that

$$\left|p^{-1}q\right|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $m \geq N_1$, we must also have

$$\left|p^{-1}q\right|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For $m \in \mathbf{Z}$, $m \geq 1$, consider

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \leq |p|_p^{-1}|q|_p^m \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p.$$

Note that, by (22),

$$\begin{aligned} \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^{m} \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\leq |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \leq |p|_p^{-1}|q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \geq |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \leq \left|p^{-1}q\right|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if $m \geq N_1$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p < \epsilon.$$

Now let $N_2 \in \mathbf{Z}$ such that

$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)N_2/(p-1)} |q|_p^{(1-\delta)N_2} < \epsilon .$$

Then we must also have

$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} < \epsilon$$

for any $m, n \in \mathbf{Z}$ such that $m \geq 0$, $n \geq 0$, and $\max\{m,n\} \geq N_2$. Let us consider

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n\right|_p \leq \left|\binom{-\xi}{m}\right|_p |q|_p^m |a_{n,\chi_m}|_p |\xi + m - 1|_p^n ,$$

where $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq 0$. For all $m \geq 0$,

$$\left|\binom{-\xi}{m}\right|_p \leq |m!|_p^{-1} r^{\delta m},$$

and by utilizing this along with (17) and (22), our expression becomes

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n\right|_p \leq |m!(n+1)!|_p^{-1} |f_\chi p|_p^{-1} r^{\delta(m+n)} |q|_p^{m+n} .$$

Since

$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n\right|_p \leq |f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} .$$

Thus if $\max\{m,n\} \geq N_2$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n\right|_p < \epsilon .$$

Let $N = \max\{N_1, N_2\}$, and let $m, n \in \mathbf{Z}$, $m \geq 0$, $n \geq -1$. Then for $\max\{m,n\} \geq N$, it must be true that

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n\right|_p < \epsilon .$$

Thus, by Proposition 2.4, the sum

$$\sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n$$

must converge. This implies that the function on the right of (21) must converge for all $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and the theorem must then hold. $\square$

Since we can now express $L_p(s,t;\chi)$ in terms of a power series in t, we can take a derivative of this function with respect to t.

LEMMA 4.6. *Let $t \in \mathbf{C}_p$, $|t|_p \le 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$\frac{\partial^n}{\partial t^n} L_p(s,t;\chi) = n! q^n \binom{-s}{n} L_p(s+n,t;\chi_n),$$

for $n \in \mathbf{Z}$, $n \ge 0$.

Proof. If $n = 0$, then the lemma is obviously true. So consider $n = 1$. Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s,t;\chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s+m;\chi_m).$$

Now,

$$m\binom{-s}{m} = -s\binom{-s-1}{m-1},$$

so that

$$\begin{aligned}\frac{\partial}{\partial t} L_p(s,t;\chi) &= \sum_{m=1}^{\infty} (-s) \binom{-s-1}{m-1} q^m t^{m-1} L_p(s+m;\chi_m) \\ &= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p\left(s+1+m;\chi_{1+m}\right) \\ &= -qs L_p(s+1,t;\chi_1).\end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s,t;\chi) = n! q^n \binom{-s}{n} L_p(s+n,t;\chi_n)$$

for some $n \in \mathbf{Z}$, $n \ge 1$. Then

$$\begin{aligned}\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s,t;\chi) &= \frac{\partial}{\partial t}\left(\frac{\partial^n}{\partial t^n} L_p(s,t;\chi)\right) \\ &= n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n,t;\chi_n).\end{aligned}$$

From the case for $n = 1$, we see that

$$\begin{aligned}n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n,t;\chi_n) &= n! q^n \binom{-s}{n} (-s-n) q L_p\left(s+n+1,t;\chi_{n+1}\right) \\ &= (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1,t;\chi_{n+1}\right).\end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s,t;\chi) = (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1,t;\chi_{n+1}\right),$$

and the lemma must hold by induction. □

With this result, we can derive a more general power series expansion of $L_p(s,t;\chi)$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \le 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m) .$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s,t;\chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s,t;\chi)$ in the form

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \beta_m (t-\alpha)^m,$$

where

$$\beta_m = \frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi)\bigg|_{t=\alpha} .$$

From Lemma 4.6

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi) = \binom{-s}{m} q^m L_p(s+m,t;\chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s+m,\alpha;\chi_m),$$

completing the proof. $\square$

4.3 RELATING $L_p(s,t;\chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s,t;\chi)$.

For the character χ, let $F_0 = \operatorname{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.