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With this result, we can derive a more general power series expansion of  $L_p(s, t; \chi)$ .

**THEOREM 4.7.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then for  $\alpha \in \mathbf{C}_p$ ,  $|\alpha|_p \leq 1$ ,*

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t - \alpha)^m L_p(s + m, \alpha; \chi_m).$$

**REMARK.** Note that Theorem 4.5 is the case of  $\alpha = 0$  here.

*Proof.* It follows from the Taylor series expansion of  $L_p(s, t; \chi)$  in the variable  $t$  about  $\alpha$  (see Proposition 2.6) that we can write  $L_p(s, t; \chi)$  in the form

$$L_p(s, t; \chi) = \sum_{m=0}^{\infty} \beta_m (t - \alpha)^m,$$

where

$$\beta_m = \frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) \Big|_{t=\alpha}.$$

From Lemma 4.6

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s, t; \chi) = \binom{-s}{m} q^m L_p(s + m, t; \chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s + m, \alpha; \chi_m),$$

completing the proof.  $\square$

#### 4.3 RELATING $L_p(s, t; \chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of  $L_p(s, t; \chi)$ .

For the character  $\chi$ , let  $F_0 = \text{lcm}(f_\chi, q)$ . Then  $f_{\chi_n} \mid F_0$  for each  $n \in \mathbf{Z}$ . Also, let  $F$  be a positive multiple of  $pq^{-1}F_0$ .

**THEOREM 4.8.** *Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then*

$$(23) \quad L_p(s, t + F; \chi) - L_p(s, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s}.$$

*Proof.* Let  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , and let  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Then from (18),

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = -\frac{1}{n} (b_n(t + F) - b_n(t)).$$

Now, (19) implies

$$\begin{aligned} b_n(t + F) - b_n(t) &= (B_{n, \chi_n}(q(t + F)) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}q(t + F))) \\ &\quad - (B_{n, \chi_n}(qt) - \chi_n(p)p^{n-1}B_{n, \chi_n}(p^{-1}qt)) \\ &= (B_{n, \chi_n}(q(t + F)) - B_{n, \chi_n}(qt)) \\ &\quad - \chi_n(p)p^{n-1} (B_{n, \chi_n}(p^{-1}q(t + F)) - B_{n, \chi_n}(p^{-1}qt)). \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} b_n(t + F) - b_n(t) &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n\chi_n(p)p^{n-1} \sum_{a=1}^{p^{-1}qF} \chi_n(a)(a + p^{-1}qt)^{n-1} \\ &= n \sum_{a=1}^{qF} \chi_n(a)(a + qt)^{n-1} - n \sum_{\substack{a=1 \\ p|a}}^{qF} \chi_n(a)(a + qt)^{n-1}. \end{aligned}$$

Therefore,

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a + qt)^{n-1}.$$

Now,  $\chi_n = \chi_1\omega^{-(n-1)}$ , so that

$$\begin{aligned} \chi_n(a)(a + qt)^{n-1} &= \chi_1(a)\omega^{-(n-1)}(a)(a + qt)^{n-1} \\ &= \chi_1(a)\langle a + qt \rangle^{n-1}. \end{aligned}$$

Thus

$$L_p(1 - n, t + F; \chi) - L_p(1 - n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a + qt \rangle^{n-1},$$

and (23) holds for all  $s = 1 - n$ , where  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all  $s$  in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable  $s$ , of the functions on the left of (23) contain  $\mathfrak{D}$ , except  $s \neq 1$  when  $\chi = 1$ . Consider now the function

$$- \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-1} \langle a + qt \rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form  $\langle a + qt \rangle^{1-s}$ , where  $a \in \mathbf{Z}$ ,  $(a, p) = 1$ , we need only show that each such function is analytic on  $\mathfrak{D}$ , and the proof will be complete.

The quantity  $\langle a + qt \rangle^{1-s}$  can be written as

$$\langle a + qt \rangle^{1-s} = \exp((1-s) \log \langle a + qt \rangle),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a + qt \rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m.$$

Since  $\langle a + qt \rangle \equiv 1 \pmod{qo}$  for  $a \in \mathbf{Z}$ ,  $(a, p) = 1$ , and  $t \in \mathbf{C}_p$ ,  $|t|_p \leq 1$ , we must also have  $\log \langle a + qt \rangle \equiv 0 \pmod{qo}$  for such  $a$  and  $t$ . Thus

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \leq \left| \frac{1}{m!} q^m (s-1)^m \right|_p$$

for all  $m$ . By (8) we can write

$$\begin{aligned} \left| \frac{1}{m!} q^m (s-1)^m \right|_p &\leq \left| p^{-m/(p-1)} q^m (s-1)^m \right|_p \\ &= \left| p^{-1/(p-1)} q (s-1) \right|_p^m. \end{aligned}$$

Thus if

$$\left| p^{-1/(p-1)} q (s-1) \right|_p < 1,$$

then

$$\left| \frac{1}{m!} (1-s)^m (\log \langle a + qt \rangle)^m \right|_p \rightarrow 0$$

as  $m \rightarrow \infty$ . So whenever  $|s-1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$ , meaning that  $s \in \mathfrak{D}$ , we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain  $\mathfrak{D}$ , except possibly for  $s = 1$  when  $\chi = 1$ , and the theorem must hold.  $\square$

COROLLARY 4.9. *Let  $s \in \mathfrak{D}$ , except  $s \neq 1$  if  $\chi = 1$ . Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \langle a \rangle^{-s}.$$

*Proof.* This follows from Theorem 4.8 since  $L_p(s, 0; \chi) = L_p(s; \chi)$  for any character  $\chi$ .  $\square$

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let  $\Delta_c$  denote the forward difference operator,  $\Delta_c x_n = x_{n+c} - x_n$ . Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that  $F_0 = \text{lcm}(f_\chi, q)$ . For  $n \in \mathbf{Z}$ ,  $n \geq 1$ , denote

$$\beta_{n,\chi}(t) = -\frac{1}{n} \left( B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}(p^{-1}qt) \right).$$

This is the polynomial structure that we utilized with respect to generalizing the  $p$ -adic  $L$ -functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either  $\chi$  or  $p$ .

THEOREM 4.10. *Let  $n$ ,  $c$ , and  $k$  be positive integers, and let  $\tau \in \mathbf{Z}_p$  such that  $|\tau|_p \leq |pq^{-1}F_0|_p$ . Then the quantity  $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$ , and, modulo  $q\mathbf{Z}_p[\chi]$ , is independent of  $n$ .*

*Proof.* Since  $\Delta_c$  is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \Delta_c^k \langle a \rangle^{n-1},$$

where  $F$  is a positive multiple of  $pq^{-1}F_0$ . Thus

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a) \langle a \rangle^{-1} \Delta_c^k \langle a \rangle^n.$$

Note that

$$(24) \quad \Delta_c^k \langle a \rangle^n = \sum_{m=0}^k \binom{k}{m} (-1)^{k-m} \langle a \rangle^{n+mc} = \langle a \rangle^n (\langle a \rangle^c - 1)^k.$$

Now,  $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$ , which implies that  $\langle a \rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$ , and thus

$$\Delta_c^k \langle a \rangle^n \equiv 0 \pmod{q^k \mathbf{Z}_p}.$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so  $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$ . Also, since  $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$ ,

$$(25) \quad q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{n-1} \left( \frac{\langle a \rangle^c - 1}{q} \right)^k$$

implies that the value of  $q^{-k} \Delta_c^k \beta_{n,\chi}(F) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$  modulo  $q\mathbf{Z}_p[\chi]$  is independent of  $n$ .

Let  $\tau \in pq^{-1}F_0\mathbf{Z}_p$ . Since the set of positive integers in  $pq^{-1}F_0\mathbf{Z}$  is dense in  $pq^{-1}F_0\mathbf{Z}_p$ , there exists a sequence  $\{\tau_i\}_{i=1}^{\infty}$  in  $pq^{-1}F_0\mathbf{Z}$ , with  $\tau_i > 0$  for each  $i$ , such that  $\tau_i \rightarrow \tau$ . Now,  $\beta_{n,\chi}(t)$  is a polynomial, which implies that  $\beta_{n,\chi}(\tau_i) \rightarrow \beta_{n,\chi}(\tau)$ . Therefore

$$\lim_{i \rightarrow \infty} (\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0)) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0).$$

The left side of this equality is 0 modulo  $q^k \mathbf{Z}_p[\chi]$ , which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]},$$

and so  $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$ . Furthermore, for  $n'$  a positive integer,

$$\begin{aligned} & \lim_{i \rightarrow \infty} ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))) \\ &= ((q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)) - (q^{-k} \Delta_c^k \beta_{n',\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n',\chi}(0))). \end{aligned}$$

Since  $\tau_i \in pq^{-1}F_0\mathbf{Z}$  for each  $i$ , the quantity on the left must also be 0 modulo  $q\mathbf{Z}_p[\chi]$ . Therefore the value of  $q^{-k} \Delta_c^k \beta_{n,\chi}(\tau) - q^{-k} \Delta_c^k \beta_{n,\chi}(0)$  modulo  $q\mathbf{Z}_p[\chi]$  is independent of  $n$ .  $\square$

**THEOREM 4.11.** *Let  $n, c, k,$  and  $k'$  be positive integers with  $k \equiv k' \pmod{p-1}$ , and let  $\tau \in \mathbf{Z}_p$  such that  $|\tau|_p \leq |pq^{-1}F_0|_p$ . Then*

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

*Proof.* Let  $k$  and  $k'$  be positive integers such that  $k \equiv k' \pmod{p-1}$ . Without loss of generality, we can assume that  $k \geq k'$ . From (25),

$$\begin{aligned} & (q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)) \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^k + \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^{k'} \\ &= - \sum_{\substack{a=1 \\ (a,p)=1}}^{q^F} \chi_1(a)\langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q}\right)^{k'} \left( \left(\frac{\langle a \rangle^c - 1}{q}\right)^{k-k'} - 1 \right), \end{aligned}$$

where  $F$  is a positive multiple of  $pq^{-1}F_0$ . If  $a$  is such that

$$\langle a \rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a \rangle^c - 1}{q}\right)^{k-k'} - 1 \equiv 0 \pmod{p\mathbf{Z}_p},$$

since  $k - k' \equiv 0 \pmod{p-1}$ . Thus

$$\begin{aligned} q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

Now let  $\tau \in pq^{-1}F_0\mathbf{Z}_p$ . Then there exists a sequence  $\{\tau_i\}_{i=1}^\infty$  in  $pq^{-1}F_0\mathbf{Z}$ , with  $\tau_i > 0$  for each  $i$ , such that  $\tau_i \rightarrow \tau$ . Consider

$$\begin{aligned} \lim_{i \rightarrow \infty} ((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))) \\ = (q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)). \end{aligned}$$

Since the left side of this equality must be 0 modulo  $p\mathbf{Z}_p[\chi]$ , the theorem must hold.  $\square$

THEOREM 4.12. Let  $n$ ,  $c$ , and  $k$  be positive integers, and let  $\tau \in \mathbf{Z}_p$  such that  $|\tau|_p \leq |pq^{-1}F_0|_p$ . Then the quantity

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo  $q\mathbf{Z}_p[\chi]$ , is independent of  $n$ .

*Proof.* We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k} L_p(1-n, F; \chi) = \binom{q^{-1}\Delta_c}{k} L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \binom{q^{-1}\Delta_c}{k} \langle a \rangle^{n-1},$$

where  $F$  is a positive multiple of  $pq^{-1}F_0$ . Then

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{-1} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n.$$

Utilizing (15), we can write

$$\begin{aligned} \binom{q^{-1}\Delta_c}{k} \langle a \rangle^n &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \Delta_c^m \langle a \rangle^n \\ &= \frac{1}{k!} \sum_{m=0}^k s(k, m) q^{-m} \langle a \rangle^n (\langle a \rangle^c - 1)^m, \end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k} \langle a \rangle^n = \langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k}.$$

Since  $q^{-1}(\langle a \rangle^c - 1) \in \mathbf{Z}_p$  for each  $a \in \mathbf{Z}$  with  $(a, p) = 1$ , we see that

$$\langle a \rangle^n \binom{q^{-1}(\langle a \rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since  $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$ , the value of this quantity modulo  $q\mathbf{Z}_p[\chi]$  is independent of  $n$ .



Now let  $\tau \in pq^{-1}F_0\mathbf{Z}_p$ , and let  $\{\tau_i\}_{i=1}^\infty$  be a sequence in  $pq^{-1}F_0\mathbf{Z}$ , with  $\tau_i > 0$  for each  $i$ , such that  $\tau_i \rightarrow \tau$ . We are working with polynomials, so that

$$\begin{aligned} \lim_{i \rightarrow \infty} \left( \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) \\ = \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0), \end{aligned}$$

which must be in  $\mathbf{Z}_p[\chi]$  since the limit of any sequence in  $\mathbf{Z}_p[\chi]$  must also be in  $\mathbf{Z}_p[\chi]$ . Now let  $n'$  be a positive integer, and consider

$$\begin{aligned} \lim_{i \rightarrow \infty} \left( \left( \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left( \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right) \\ = \left( \left( \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0) \right) - \left( \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n',\chi}(0) \right) \right). \end{aligned}$$

The quantity on the left must be 0 modulo  $q\mathbf{Z}_p[\chi]$ , which implies that the value of

$$\binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k} \beta_{n,\chi}(0)$$

modulo  $q\mathbf{Z}_p[\chi]$  is independent of  $n$ .  $\square$

#### 4.4 GENERALIZED BERNOULLI POWER SERIES

In [9] we find a definition of ordinary Bernoulli numbers of negative index,  $B_{-n}$ , where  $n \in \mathbf{Z}$ ,  $n \geq 1$ , in the field  $\mathbf{Q}_p$ , given by

$$(26) \quad B_{-n} = \lim_{k \rightarrow \infty} B_{\phi(p^k)-n},$$

where the limit is taken in a  $p$ -adic sense. Note that  $\phi(p^k) \rightarrow 0$  in  $\mathbf{Z}_p$  as  $k \rightarrow \infty$ . Since  $|B_m|_p$  is bounded for all  $m \in \mathbf{Z}$ ,  $m \geq 0$ , we must have

$$\begin{aligned} B_{-n} &= \lim_{k \rightarrow \infty} \left( 1 - p^{\phi(p^k)-n-1} \right) B_{\phi(p^k)-n} \\ &= \lim_{k \rightarrow \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n); \omega^{-n}) \\ &= nL_p(n + 1; \omega^{-n}). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that  $B_m = 0$  for any odd  $m \in \mathbf{Z}$ ,  $m \geq 3$ . Thus (26) implies that  $B_{-n} = 0$  for any odd  $n \in \mathbf{Z}$ ,  $n \geq 1$ . Furthermore, we have the following: