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Artikel: ORDERINGS OF MAPPING CLASS GROUPS AFTER THURSTON
Autor: SHORT, Hamish / WIEST, Bert
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and $1_{B_n} < \varphi$. In this ordering the commutator subgroup is convex [19], and we leave it to the reader to verify that no Thurston-type ordering has this property.

3. MAIN RESULTS

We shall mainly be interested in the case $S = D_n$ $(n \geq 2)$, where D_n is the closed unit disk in $\mathbf{C}$, with n punctures lined up in the real interval $(-1, 1)$; in this case the mapping class group is a braid group: $\mathcal{MCG}(D_n) = B_n$. We recall that for $\alpha \in (0, \pi)$ we denote by γ_α the geodesic which starts at the basepoint with angle α with ∂S, and by $\widetilde{\gamma}_\alpha$ its preimage in the universal cover starting at the basepoint of $S^\sim$.

DEFINITION 3.1. A geodesic γ_α, $\alpha \in (0, \pi)$, is said to be of *finite type* if it satisfies at least one of the following conditions:

(a) there exists a finite initial segment γ_α^t such that any two punctures that lie in the same path component of $S \backslash \gamma_\alpha^t$ also lie in the same path component of $S \backslash \gamma_\alpha$, or

(b) it falls into a puncture, or

(c) it spirals towards a simple closed geodesic.

If a geodesic γ_α is not of finite type then we say it is of *infinite type*. We also define the ordering of $\mathcal{MCG}(S)$ induced by a geodesic γ_α to be of finite or infinite type if γ_α is of finite or infinite type.

An infinite type geodesic looks as follows. All its self intersections occur in some finite initial segment γ_α^t. At least one of the path components of $S \backslash \gamma_\alpha^t$ contains three or more punctures in its interior, and the closure of $\gamma_\alpha \backslash \gamma_\alpha^t$ is a geodesic lamination without closed leaves inside such a component. In particular, there is a pair of punctures which are separated by the whole geodesic, but not by any finite initial segment. (Note that the geodesic $\gamma_\alpha \backslash \gamma_\alpha^t$ is isolated from both sides – in this it is very different from leaves of geodesic laminations on surfaces without boundary.)

DEFINITION 3.2. For a geodesic γ_α of finite respectively infinite type we say that it *fills the surface in finite* respectively *infinite time* if all punctures lie in different path components of $S \backslash \gamma_\alpha$. By contrast, a geodesic γ_α *does not fill* the surface if $S \backslash \gamma_\alpha$ has a path component that contains two punctures.

The aim of the rest of the paper is to prove the following theorems. Recall that every point $\alpha \in (0, \pi)$ gives rise to a – possibly partial – ordering of $\mathcal{MCG}(S)$. The first theorem gives criteria for these orderings to be total or, equivalently, for the orbit of α to be free.

THEOREM 3.3. *Let S be any hyperbolic surface.*

(a) *If a geodesic γ_α does not fill S, then the orbit of $\alpha \in (0, \pi)$ is not free.*

(b) *If γ_α is of finite type, then the converse holds as well: if γ_α fills the surface, then α has free orbit.*

(c) *Let $\mathcal{I} := \{\alpha \mid \gamma_\alpha$ is of infinite type$\} \subset (0, \pi)$. Then $\mathcal{I}$ is uncountable, and all but countably many of its elements have free orbits. In any neighbourhood of an $\alpha \in \mathcal{I}$ there exist points of both finite and infinite type, i.e. there are $\alpha' \neq \alpha$ and $\beta \in (0, \pi)$ such that $\gamma_{\alpha'} \in \mathcal{I}$ and $\gamma_\beta \notin \mathcal{I}$.*

The next theorem gives a classification of orders of Thurston-type.

THEOREM 3.4. *If S is a punctured disk, we have:*

(a) *An ordering cannot be both of finite and infinite type.*

(b) *Given two geodesics $\gamma_\alpha, \gamma_\beta$ of finite type, one can decide whether or not they determine the same ordering.*

(c) *Given two geodesics $\gamma_\alpha, \gamma_\beta$ of infinite type, one can decide whether or not they determine the same ordering. For instance, if γ_α and γ_β are embedded, then they determine the same ordering if and only if $\beta = \Delta^{2k}(\alpha)$ for some $k \in \mathbf{Z}$ (i.e. if γ_β is obtained from γ_α by sliding the starting point $2k$ times around ∂D_n).*

(Note that part (a) is not immediately clear: it is conceivable that finite and infinite type geodesics induce the same orderings.) In fact, we shall develop machinery which gives a very good and explicit understanding of finite type orderings:

THEOREM 3.5. *There are only finitely many conjugacy classes of orderings of finite type of $\mathcal{MCG}(D_n) = B_n$. The number N_n of conjugacy classes can be calculated by the following recursive formula*

$$N_2 = 1 \quad \textit{and} \quad N_n = N_{n-1} + \sum_{k=2}^{n-2} \binom{n-2}{k-1} N_k \, N_{n-k} \, .$$

We do not know if there exists a "closed" formula for N_n. The following list gives the first few values:

n	2	3	4	5	6	7	8
N_n	1	1	3	9	39	189	1197

Theorems 3.4 and 3.5 almost certainly generalise to mapping class groups of other negatively curved surfaces, but in order to keep our machinery simple, we stick to the special case of punctured disks.

4. ORDERINGS OF MAPPING CLASS GROUPS USING CURVE DIAGRAMS

In this section we present another method for constructing left orderings on B_n, using certain diagrams on D_n, which we call *curve diagrams*. Both the definition of curve diagrams and the orderings associated to them are generalisations of similar concepts in [9].

CONVENTION. Whenever we talk about geodesics in D_n, we think of the punctures as being holes in the disk, whose neighbourhoods on the disk have the geometry of cusps. By contrast, when we talk about curve diagrams, we think of the punctures as distinguished points on, and belonging to, the disk, and we ignore the geometric structure. This changing perspective should not cause confusion.

DEFINITION 4.1. A (partial) curve diagram Γ is a diagram on D_n consisting of $j \leqslant n-1$ closed, oriented arcs which are labelled $\Gamma_1, \ldots, \Gamma_j$. Moreover, the boundary circle of D_n is labelled Γ_0, and by abuse of notation we shall refer to it as an "arc" of Γ. We require:

(1) every path component of $D_n \backslash \Gamma$ has at least one puncture in its interior,

(2) $\bigcup_{i=0}^{j} int(\Gamma_i)$ is embedded and disjoint from the punctures (where *int* denotes the interior),

(3) the starting point of the i^{th} arc lies in $\bigcup_{k=0}^{i-1} \Gamma_k$, i.e. on one of the previous arcs,

(4) the end point of the i^{th} arc lies in one of the previous arcs, or on an earlier point of the i^{th} arc, or in a puncture.

In the special case that $j = n-1$, so that in (1) every path component contains precisely one puncture, we say Γ is a total curve diagram.