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Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
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COROLLARY 7. *If $\gamma = e(\rho)$, for some $\rho \in H^2(\Gamma, \mathbf{R})$, then the subgroup of* $\mathbf{R}$, $\Delta = \mathrm{tr}_\Gamma(K_0(C^*_r(\Gamma,\gamma)))$ *contains the group:*

$$\langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho) \rangle .$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho)\rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let Γ be the fundamental group of a compact Riemann surface of positive genus, $\gamma \in H^2(\Gamma, S^1)$ be a 2-cocycle and $\theta \in \mathbf{R}/\mathbf{Z}$ the class of γ in $H^2(\Gamma,\mathbf{R})/H^2(\Gamma,\mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. Then the image of $K_0(C^*_r(\Gamma,\gamma))$ by the canonical trace $\zeta = \mathrm{Tr}_\Gamma$ is equal to the subgroup $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.*

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C^*_r(\Gamma,\gamma)$ (for any value of γ), one gets that the corresponding C^*-algebras are isomorphic only when the Γ's are the same (using K_1) and when the γ's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞-manifold, and let F be a C^∞-foliation of V. Thus F is a C^∞-integrable sub-vector bundle of TV. As in [33] let G be the holonomy groupoid (graph) of (V,F). The manifold V is assumed to be Hausdorff and second countable. G, however, is a C^∞-manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞-paths

$$\gamma\colon [0,1] \to V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0,1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \underset{r}{\overset{s}{\rightrightarrows}} V$.

Let Z be a possibly non-Hausdorff C^∞-manifold. Assume given a C^∞-map $\rho\colon Z \to V$, set

$$Z \circ G = \{(z,\gamma) \in Z \times G \mid \rho(z) = s(\gamma)\} .$$

A C^∞ *right action of G on Z* is a C^∞-map

$$Z \circ G \to Z$$

denoted by

$$(z, \gamma) \to z\gamma$$

such that

$$\rho(z\gamma) = r(\gamma)\,, \quad (z\gamma)\gamma' = z(\gamma\gamma')\,, \quad (zl_p) = z\,,$$

where l_p denotes the constant path at $p \in V$.

An action of G on Z is *proper* if:

(i) the map $Z \circ G \to Z \times Z$ given by $(z, \gamma) \mapsto (z, z\gamma)$ is proper (i.e. the inverse image of a compact set is compact);

(ii) the quotient space Z/Γ is Hausdorff. Here Z/Γ is the set of equivalence classes of $z \in Z$ where $z \sim z'$ if, for some $\gamma \in G$, $z\gamma = z'$.

Specializing to $Z = V$, the groupoid G acts on V by $\rho(p) = p$ and

$$p\gamma = \gamma(1)$$

($p \in V$, $\gamma \in G$, $p = \gamma(0)$). For many examples this action of G on V is not proper. Set $\nu_p = T_pV/F_p$, so that ν is the normal bundle of the foliation. ν is a G-vector bundle since the derivative of holonomy gives a linear map

$$\nu_p \mapsto \nu_{p\gamma}\,.$$

This is, of course, just the well-known fact that ν is flat along the leaves of the foliation.

More generally, if Z is a G-manifold, then the orbits of the G-action foliate Z. Denote the normal bundle of this foliation by $\widetilde{\nu}$. Then $\widetilde{\nu}$ is a G-vector bundle on Z.

If Z is a proper G-manifold, a G-vector bundle on Z with G-compact support is a triple (E_0, E_1, σ) where E_0, E_1 are G-vector bundles on Z and $\sigma\colon E_0 \to E_1$ is a morphism of G-vector bundles with Support(σ) G-compact. As in §2 above one then defines $V^i_G(Z)$ and $K^i_G(Z)$, $i = 0, 1$. These are defined and used *only* for proper G-manifolds.

DEFINITION 1. A *K-cocycle for* (V, F) is a pair (Z, ξ) such that

(1) Z is a proper G-manifold,

(2) $\xi \in V^*_G[(\widetilde{\nu})^* \oplus \rho^*\nu^*]$, where $\rho\colon Z \to V$ is given by the action of G on Z.

In [12] and [14] a canonical C^*-algebra $C^*(V, F)$ is constructed. This C^*-algebra can heuristically be thought of (up to Morita equivalence) as the algebra of continuous functions on the "space of leaves" of the foliation. Thus $K_*C^*(V,F)$ can be viewed as the K-theory of the "space of leaves" of the foliation.

To define the geometric K-theory $K^*(V,F)$ we proceed quite analogously to §2 above.

THEOREM 2. *Let (Z,ξ) be a cocycle for (V,F). Then (Z,ξ) determines an element in $K_*C^*(V,F)$.*

Proof. If $\rho\colon Z \to V$ is a submersion then ξ gives rise to the symbol of a G-equivariant family of elliptic operators D, parametrized by the points of V. The K-theory index of this family D is the desired element of $K_*C^*(V,F)$.

If $\rho\colon Z \to V$ is not a submersion, then as in the proof of Theorem 1 of §2 one reduces to the submersion case. □

REMARK 3. With D as in the proof of the Theorem, Index$(D) \in K_*C^*(V,F)$ will be denoted $\mu(Z,\xi)$. For $\xi \in V_G^i[(\widetilde{\nu})^* \oplus \rho^*\nu^*]$, $\mu(Z,\xi) \in K_i C^*(V,F)$, $i = 0,1$.

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ & {\scriptstyle\rho_1}\searrow \quad \swarrow{\scriptstyle\rho_2} & \\ & V & \end{array}$$

where Z_1, Z_2 are G-manifolds with Z_1, Z_2 proper and h is a G-map. There is then a Gysin map

$$h_!\colon K_G^i[(\widetilde{\nu}_1)^* \oplus \rho_1^*\nu^*] \to K_G^i[(\widetilde{\nu}_2)^* \oplus \rho_2^*\nu^*].$$

THEOREM 4. *If $\xi_1 \in V_G^*[(\widetilde{\nu}_1)^* \oplus \rho_1^*\nu]$ then $\mu(Z_1,\xi_1) = \mu(Z_2, h_!(\xi_1))$.*

REMARK 5. Let $\Gamma(V,F)$ be the collection of all K-cocycles (Z,ξ) for (V,F). On $\Gamma(V,F)$ impose the equivalence relation $\sim$, where $(Z,\xi) \sim (Z',\xi')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z'' & \xleftarrow{h'} & Z' \\ & {\scriptstyle\rho}\searrow & \downarrow{\scriptstyle\rho''} & \swarrow{\scriptstyle\rho'} & \\ & & V & & \end{array}$$

with h and h' G-maps and with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 6. $K^*(V,F) = \Gamma(V,F)/\sim$. Addition in $K^*(V,F)$ is by disjoint union of K-cocycles. The natural homomorphism of abelian groups

$$K^i(V,F) \to K_i\, C^*(V,F)$$

is defined by

$$(Z,\xi) \to \mu(Z,\xi).$$

CONJECTURE. *$\mu: K^*(V,F) \to K_* C^*(V,F)$ is an isomorphism.*

REMARK 7. Calculations of M. Pennington [25] and A.M. Torpe [32] verify the conjecture for certain foliations.

Given (V,F), let BG be the classifying space of the holonomy groupoid G. Since ν is a G-vector bundle on V, ν induces a vector bundle τ on BG. As in §3 above there is then a natural map

$$K_*^\tau(BG) \to K^*(V,F).$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \to K^*(V,F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \to K^*(V,F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \to K^*(V,F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu: K^*(V,F) \to K_* C^*(V,F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^*-algebra. If G is a Lie group and X is a G-manifold, then using A as coefficients there is both a geometric and an analytic K-theory for (X,G). The analytic K-theory is the K-theory of the C^*-algebra $(C_0(X) \rtimes G) \otimes A$.