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Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS

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DEFINITION 6. $K^*(V,F) = \Gamma(V,F)/\sim$. Addition in $K^*(V,F)$ is by disjoint union of K-cocycles. The natural homomorphism of abelian groups

$$K^i(V,F) \to K_i\, C^*(V,F)$$

is defined by

$$(Z,\xi) \to \mu(Z,\xi)\,.$$

CONJECTURE. *$\mu\colon K^*(V,F) \to K_* C^*(V,F)$ is an isomorphism.*

REMARK 7. Calculations of M. Pennington [25] and A.M. Torpe [32] verify the conjecture for certain foliations.

Given (V,F), let BG be the classifying space of the holonomy groupoid G. Since ν is a G-vector bundle on V, ν induces a vector bundle τ on BG. As in §3 above there is then a natural map

$$K_*^\tau(BG) \to K^*(V,F)\,.$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \to K^*(V,F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \to K^*(V,F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \to K^*(V,F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu\colon K^*(V,F) \to K_* C^*(V,F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^*-algebra. If G is a Lie group and X is a G-manifold, then using A as coefficients there is both a geometric and an analytic K-theory for (X,G). The analytic K-theory is the K-theory of the C^*-algebra $(C_0(X) \rtimes G) \otimes A$.

The geometric K-theory is obtained from K-cocycles (Z,ξ,f) where Z,f are as in §2 and $\xi = \{E_0 \xrightarrow{\sigma} E_1\}$ uses G-vector bundles E_0, E_1 on $T^*Z \oplus f^*T^*X$ such that the fibres of E_i are finitely generated projective modules over A. Denote this geometric K-theory by $K^*(X,G;A)$. The natural map

$$K^i(X,G;A) \to K_i[(C_0(X) \rtimes G) \otimes A]$$

is defined by using elliptic operators in the spirit of Miscenko-Fomenko [22]. We conjecture that this natural map is an isomorphism.

In the notation of Kasparov [18] the group denoted here by $K_*[C_0(X) \rtimes G]$ is $KK(\mathbf{C}, C_0(X) \rtimes G)$. For the K-homology group $KK(C_0(X) \rtimes G, \mathbf{C})$ there is a geometric group $K_*(X,G)$ which is the G-equivariant version of the topologically defined K-homology of [9]. Using transversally elliptic operators [2] one then obtains a natural map

$$K_*(X,G) \to KK(C_0(X) \rtimes G, \mathbf{C}).$$

We conjecture that this map is injective and that its image is dense (with respect to the natural topology) in $KK(C_0(X) \rtimes G, \mathbf{C})$.

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Proposition 1 (A. Borel [10]). *Let G be a Lie group with $\pi_0 G$ finite and maximal compact subgroup H. If Z is any proper G-manifold then there exists a G-map from Z to $H \backslash G$.*

In §5 above this was proved for G a connected semi-simple Lie group with finite center. By the argument of §5, Borel's result implies:

Corollary 2. *Let G be a Lie group with $\pi_0 G$ finite. For any G-manifold X there is an isomorphism of abelian groups*

$$K^i_H(X \times (\mathfrak{h}\backslash\mathfrak{g})^*) \to K^i(X,G) \qquad (i = 0,1).$$

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