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DEFINITION 6.  $K^*(V, F) = \Gamma(V, F)/\sim$ . Addition in  $K^*(V, F)$  is by disjoint union of  $K$ -cocycles. The natural homomorphism of abelian groups

$$K^i(V, F) \rightarrow K_i C^*(V, F)$$

is defined by

$$(Z, \xi) \rightarrow \mu(Z, \xi).$$

CONJECTURE.  $\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$  is an isomorphism.

REMARK 7. Calculations of M. Pennington [25] and A. M. Torpe [32] verify the conjecture for certain foliations.

Given  $(V, F)$ , let  $BG$  be the classifying space of the holonomy groupoid  $G$ . Since  $\nu$  is a  $G$ -vector bundle on  $V$ ,  $\nu$  induces a vector bundle  $\tau$  on  $BG$ . As in §3 above there is then a natural map

$$K_*^\tau(BG) \rightarrow K^*(V, F).$$

PROPOSITION 8. *The natural map  $K_*^\tau(BG) \rightarrow K^*(V, F)$  is rationally injective. If  $G$  is torsion free then  $K_*^\tau(BG) \rightarrow K^*(V, F)$  is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map  $K_*^\tau(BG) \rightarrow K^*(V, F)$  may fail to be an isomorphism.

THEOREM 10. *If  $F$  admits a  $C^\infty$  Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu: K^*(V, F) \rightarrow K_* C^*(V, F)$$

*is injective.*

## 10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let  $A$  be a  $C^*$ -algebra. If  $G$  is a Lie group and  $X$  is a  $G$ -manifold, then using  $A$  as coefficients there is both a geometric and an analytic  $K$ -theory for  $(X, G)$ . The analytic  $K$ -theory is the  $K$ -theory of the  $C^*$ -algebra  $(C_0(X) \rtimes G) \otimes A$ .

The geometric  $K$ -theory is obtained from  $K$ -cocycles  $(Z, \xi, f)$  where  $Z, f$  are as in §2 and  $\xi = \{E_0 \xrightarrow{\sigma} E_1\}$  uses  $G$ -vector bundles  $E_0, E_1$  on  $T^*Z \oplus f^*T^*X$  such that the fibres of  $E_i$  are finitely generated projective modules over  $A$ . Denote this geometric  $K$ -theory by  $K^*(X, G; A)$ . The natural map

$$K^i(X, G; A) \rightarrow K_i[(C_0(X) \rtimes G) \otimes A]$$

is defined by using elliptic operators in the spirit of Miscenko-Fomenko [22]. We conjecture that this natural map is an isomorphism.

In the notation of Kasparov [18] the group denoted here by  $K_*[C_0(X) \rtimes G]$  is  $KK(\mathbf{C}, C_0(X) \rtimes G)$ . For the  $K$ -homology group  $KK(C_0(X) \rtimes G, \mathbf{C})$  there is a geometric group  $K_*(X, G)$  which is the  $G$ -equivariant version of the topologically defined  $K$ -homology of [9]. Using transversally elliptic operators [2] one then obtains a natural map

$$K_*(X, G) \rightarrow KK(C_0(X) \rtimes G, \mathbf{C}).$$

We conjecture that this map is injective and that its image is dense (with respect to the natural topology) in  $KK(C_0(X) \rtimes G, \mathbf{C})$ .

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**PROPOSITION 1** (A. Borel [10]). *Let  $G$  be a Lie group with  $\pi_0 G$  finite and maximal compact subgroup  $H$ . If  $Z$  is any proper  $G$ -manifold then there exists a  $G$ -map from  $Z$  to  $H \backslash G$ .*

In §5 above this was proved for  $G$  a connected semi-simple Lie group with finite center. By the argument of §5, Borel's result implies:

**COROLLARY 2.** *Let  $G$  be a Lie group with  $\pi_0 G$  finite. For any  $G$ -manifold  $X$  there is an isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h} \backslash \mathfrak{g})^*) \rightarrow K^i(X, G) \quad (i = 0, 1).$$

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