

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 46 (2000)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY  
**Autor:** Chatterji, Srishti D.  
**Kapitel:** §1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-64804>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 02.04.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## REMARKS ON THE HAUSDORFF-YOUNG INEQUALITY

by Srishti D. CHATTERJI

### §1. INTRODUCTION

A standard version of the Hausdorff-Young inequality for a locally compact commutative group  $G$  can be given as follows: for a fixed Haar measure in  $G$ , let  $f \in L^1(G) \cap L^2(G)$ ; if  $1 \leq p \leq 2$ ,  $p' = p/(p - 1)$ , then

$$(1) \quad \|\widehat{f}\|_{p'} \leq \|f\|_p$$

where

$$(2) \quad \widehat{f}(\gamma) = \int_G f(x) \overline{\gamma(x)} dx, \quad \gamma \in \widehat{G},$$

$\widehat{G}$  being the dual group of  $G$ , endowed with a Haar measure which is such that for  $p = p' = 2$ , there is equality in (1); that this last condition can be met is one form of Plancherel's theorem in  $L^2(G)$ . Note that, for  $1 \leq p \leq 2$ ,  $\|f\|_p < \infty$  if  $f$  is in  $L^1(G) \cap L^2(G)$ , the latter space being dense in each  $L^p(G)$ ,  $1 \leq p \leq 2$ . Hence, because of the Hausdorff-Young inequality (1), the Fourier transform  $\mathcal{F}_p f$  can be defined uniquely for all  $f \in L^p(G)$ ,  $1 \leq p \leq 2$ , in such a way that

$$(3) \quad \mathcal{F}_p: L^p(G) \rightarrow L^{p'}(\widehat{G})$$

is a linear contraction with  $\mathcal{F}_p f = \widehat{f}$  for all  $f$  in  $L^1(G) \cap L^2(G)$ . It is known that, for each  $p \in [1, 2]$ ,  $\mathcal{F}_p$  is injective and that if  $f \in L^{p_1}(G) \cap L^{p_2}(G)$ ,  $1 \leq p_1, p_2 \leq 2$ , then  $\mathcal{F}_{p_1} f = \mathcal{F}_{p_2} f$  a.e. on  $\widehat{G}$ ; see [HR] vol.2, chap. VIII ((31.26), p.229; (31.31), p.231). The purpose of the present note is to prove

(Thm. 1) by a very simple general argument that the operator  $\mathcal{F}_p$  in (3) is surjective only in the following obvious cases: (i)  $p = p' = 2$  or (ii)  $G$  finite. This fact is now well-known ([HR] vol. 2, p. 227, pp. 430–431); however, most of the known proofs of this depend on a careful analysis of the group  $G$  whereas our proof shows that this is an immediate consequence of a general theorem concerning the isomorphism of arbitrary  $L^p$ -spaces (stated in §2). From this we deduce fairly simply that for any infinite locally compact commutative group  $G$ , the inequality (1) cannot be extended to the case  $2 < p < \infty$ ; the exact statement is given as Thm. 2 in §3. I have not seen this statement given in complete generality elsewhere, although it is highly likely to be known to many.

We set up the necessary notations in §2, state and prove the facts alluded to above in §3 and add a few historical comments in §4; a short appendix (§5) is added to explain the  $L^p$ -isomorphism theorem stated in §2.

We have not tried to extend our theorems to the case of  $G$  non-commutative, using for  $\widehat{G}$  the set of all equivalence classes of continuous unitary irreducible representations of  $G$ . For  $G$  compact, this has been done (for our Thm. 1) in [HR] vol. 2, (37.19), p. 429; our analysis carries over to this case as well without any difficulty. However, we have preferred to leave out the non-commutative case entirely in this paper, except to make a few remarks on it in §4.

## §2. NOTATIONS AND SOME KNOWN FACTS

Our reference for general functional analysis and integration theory is [DS] and that for group theory is [HR]. A measure space is a triple  $(X, \Sigma, \mu)$  where  $\Sigma$  is a  $\sigma$ -algebra of subsets of the abstract set  $X$  and  $\mu: \Sigma \rightarrow [0, \infty]$  is a  $\sigma$ -additive positive measure; no finiteness or  $\sigma$ -finiteness conditions will be imposed a priori on  $\mu$ . Then  $L^p(\mu)$ ,  $1 \leq p \leq \infty$ , will denote the usual Banach space associated with  $\Sigma$ -measurable complex-valued functions  $f$  defined on  $X$  with  $\|f\|_p < \infty$ ,  $\|f\|_p$  being the standard  $L^p$ -norm with respect to  $\mu$ . If  $G$  is a locally compact commutative group (always supposed to be Hausdorff),  $L^p(G)$ ,  $1 \leq p \leq \infty$ , will stand for the associated  $L^p$ -space obtained by fixing some Haar (invariant) measure on  $G$ , and  $\widehat{G}$  will stand for the dual group, formed by the continuous homomorphisms (characters)

$$\gamma: G \rightarrow \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}.$$