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5.1. LEMMA. Let  $N$  be a norm of degree  $n$  on a  $k$ -algebra  $A$ . For each element  $\alpha$  of  $A$  the characteristic polynomial  $P_\alpha^N(t) = N_{k[t]}(t - \alpha)$  is monic of degree  $n$ .

Moreover, the trace  $\text{Tr}^N$  is a  $k$ -linear map  $A \rightarrow k$ .

*Proof.* Let  $s, t, u, v$  be independent variables over the ring  $k$ . For each element  $\beta$  in  $A$  the norm  $N_{k[s,t,u]}(t - \alpha s - \beta u)$  is a polynomial in  $k[s, t, u]$ . Since  $N$  is of degree  $n$  we have that  $N_{k[s,t,u,v]}(vt - \alpha vs - \beta vu) = v^n N_{k[s,t,u]}(t - \alpha s - \beta u)$ . It follows that  $N_{k[s,t,u]}(t - \alpha s - \beta u)$  is homogeneous of degree  $n$  in  $k[s, t, u]$ . In particular the coefficient of  $t^{n-1}$  is of the form  $as + bu$  with  $a$  and  $b$  in  $k$ . By evaluating the polynomial  $N_{k[s,t,u]}(t - \alpha s - \beta u)$  at  $s = 0, u = 0$ , it follows that the coefficient to  $t^n$  is equal to 1. Hence  $N_{k[t]}(t - \alpha)$  is a monic polynomial of degree  $n$ , and  $a = -\text{Tr}^N(\alpha)$ . Similarly,  $b = -\text{Tr}^N(\beta)$ . Hence we have that  $\text{Tr}^N(\alpha s + \beta u) = -(as + bu) = \text{Tr}^N(\alpha)s + \text{Tr}^N(\beta)t$ . Specializing  $s$  and  $t$  to any pair of elements of  $k$  the second part of the Lemma follows.  $\square$

5.2. EXAMPLE. Let  $M$  be a free module of rank  $n$  over  $k$ , or more generally a projective  $k$ -module of constant rank  $n$ . Then the determinant defines a norm of degree  $n$  on  $\text{End}_k(M)$ .

Let  $A$  be a  $k$ -algebra which is free of rank  $n$  as a  $k$ -module. Left multiplication by elements of  $A$  define an injection  $A \rightarrow \text{End}_k(A)$  of  $k$ -algebras. By restriction we obtain a norm of degree  $n$  on  $A$ .

## 6. NORMS AND RESULTANTS

Let  $F(x) = f_0 + \cdots + f_m x^m$  and  $P(x) = p_0 + \cdots + p_n x^n$  be polynomials of degree  $m$ , respectively  $n$  in the  $k$ -algebra  $k[x]$  of polynomials in the variable  $x$  with coefficients in  $k$ . The *resultant*  $\text{Res}(F, P)$  of  $F$  and  $P$  is the determinant of the  $(m+n) \times (m+n)$ -matrix  $D(F, P)$  whose columns are the coefficients of the polynomials  $F, xF, \dots, x^{n-1}F, P, xP, \dots, x^{m-1}P$ . Note that the definition is asymmetric in  $F$  and  $P$  in the sense that  $\text{Res}(F, P) = (-1)^{mn} \text{Res}(P, F)$ .

When  $P$  is monic the resultant is equal to the determinant of the endomorphism induced by multiplication by  $F$  on the free  $k$ -module  $k[x]/(P(x))$  of rank  $n$ . To see this we note that for  $i = 0, \dots, n-1$  we can write  $x^i F = Q_i P + R_i$  in  $k[x]$ , where  $Q_i(x)$  and  $R_i(x)$  are of degrees at most  $m-1$ , respectively  $n-1$ . It follows that the determinant of  $D(F, P)$  is equal to the determinant of the  $(m+n) \times (m+n)$ -matrix  $B(F, P)$  whose columns are the

coefficients of the polynomials  $R_0, \dots, R_{n-1}, P, xP, \dots, x^{m-1}P$ . We see that the  $n \times n$ -block  $C(F, P)$  in the upper left corner of  $B(F, P)$  is the matrix  $C(F, P)$  of the map induced by multiplication by  $F$  on  $k[x]/(P(x))$ , and the  $m \times m$ -block in the lower right corner is upper triangular with 1's on the diagonal. Moreover, the entries of  $C(F, P)$  are the only non-zero entries in the first  $n$  columns of  $B(F, P)$ . It follows that  $\text{Res}(F, P) = \det C(F, P)$ , as we claimed.

6.1. EXAMPLE. When  $P$  is a monic polynomial we saw in Example 5.2 that the  $k$ -algebra  $k[x]/(P(x))$  which is free of rank  $n$  as a  $k$ -module has a canonical norm. Via the canonical map  $k[x] \rightarrow k[x]/(P(x))$  we obtain a canonical norm  $N'_P$  on  $k[x]$ . The above interpretation of the resultant can then be written as

$$(6.1.1) \quad (N'_P)_R(F) = \text{Res}(F, P)$$

for all commutative  $k$ -algebras  $R$  and all polynomials  $F(x)$  in  $R[x] = R \otimes_k k[x]$ . By an easy computation of the determinant defining  $\text{Res}(t - x, P)$ , we obtain that the characteristic polynomial of  $x$  with respect to  $N'_P$  is

$$P_x^{N'_P}(t) = P(t).$$

6.2. EXAMPLE. We shall introduce a second important norm on  $k[x]$ . Let  $P(x)$  be a monic polynomial of degree  $n$  in the  $k$ -algebra  $k[x]$ . There is a canonical ring extension  $k \subseteq k' = k[\lambda_1, \dots, \lambda_n]$  such that  $P(x)$  splits as  $P(x) = \prod_{i=1}^n (x - \lambda_i)$  in  $k'[x]$ . The extension is obtained by induction starting with  $k = k_0$  and  $P_0(x) = P(x)$ , and constructing  $k_i = k[\lambda_1, \dots, \lambda_i]$  and  $P_i(x) \in k[\lambda_1, \dots, \lambda_i][x]$  from  $k_{i-1}$  and  $P_{i-1}$ , for  $i = 1, 2, \dots, n$ , by  $k_i = k_{i-1}[x]/(P_{i-1}(x)) = k_{i-1}[\lambda_i]$ , where  $\lambda_i$  is the class of  $x$ , and by  $P_i(x) = P_{i-1}(x)/(x - \lambda_i)$ . We note that  $k'$  is a free  $k$ -module of rank  $n!$ . The algebra  $k'$  is sometimes called the *universal decomposition algebra* for  $P$  (see [B1], §6, p. 68).

For every commutative  $k$ -algebra  $R$  and every polynomial  $G$  in  $R[x] = R \otimes_k k[x]$  we have that  $\prod_{i=1}^n G(\lambda_i)$  is symmetric in  $\lambda_1, \dots, \lambda_n$ , and consequently lies in the image of the inclusion  $R \subseteq k' \otimes_k R$ . We obtain a map  $(N''_P)_R: R \otimes_k k[x] \rightarrow R$  defined by  $(N''_P)_R(G) = \prod_{i=1}^n G(\lambda_i)$ . In this way we obtain a norm  $N''_P$  of degree  $n$  on  $k[x]$  and the characteristic polynomial of  $x$  with respect to the norm  $N''_P$  is

$$P_x^{N''_P}(t) = \prod_{i=1}^n (t - \lambda_i) = P(t).$$