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# THE WITT GROUP OF LAURENT POLYNOMIALS

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ABSTRACT. We give a direct, self-contained proof of the fact that for a large class of rings $A$, in particular for all regular rings with involution, $W(A[t,1/t]) = W(A) \oplus W(A)$.

## 1. INTRODUCTION

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring $A$. These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on $A$ and for their proofs we will use nothing more than a general result on the $K$-theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on $A$ are necessary.

We begin by recalling briefly some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let $A$ be an associative ring with an involution denoted by $a \mapsto a^\circ$. Except in §2 we will always assume that 2 is invertible in $A$. If $M$ is a right $A$-module, we denote by $M^*$ its dual $\mathrm{Hom}_A(M,A)$ endowed with the right action of $A$ given by $fa(x) = a^\circ f(x)$ for any $f\colon M \to A$ and $a \in A$. If $P$ is a finitely generated projective right $A$-module we identify it with $P^{**}$ through the canonical isomorphism mapping $x \in P$ to $\widehat{x}\colon P^* \to A$ defined by $\widehat{x}(f) = f(x)$.

Let $\epsilon$ be 1 or $-1$. An *$\epsilon$-hermitian space* over $A$ is a pair $(P,\alpha)$ consisting of a finitely generated projective right $A$-module $P$ and an $A$-isomorphism $\alpha\colon P \to P^*$ satisfying $\alpha = \epsilon\alpha^*$. For brevity $\epsilon$-hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring $A$) is also called a *quadratic space*.

Two spaces $(P, \alpha)$ and $(Q, \beta)$ are *isometric* if there exists an $A$-isomorphism $\varphi: P \to Q$ such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P^* & \xleftarrow[\varphi^*]{} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left(P \oplus P^*, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)\right).$$

The *orthogonal sum* of two spaces $(P, \alpha)$ and $(Q, \beta)$ is the space

$$(P, \alpha) \perp (Q, \beta) = (P \oplus Q, \alpha \oplus \beta).$$

If $(P, \alpha)$ is a space and $M$ a submodule of $P$ we denote by $M^\perp$ the orthogonal of $M$, defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^*,$$

where $i^*$ is the dual of the inclusion $i: M \to P$. A submodule $M$ of $P$ is *totally isotropic* if $M \subseteq M^\perp$. A *sublagrangian* of a space $(P, \alpha)$ is a totally isotropic direct factor of $P$. A *lagrangian* of $(P, \alpha)$ is a sublagrangian $L$ such that $L = L^\perp$. For instance, $P$ and $P^*$ are lagrangians of $H(P)$.

The Witt group $W(A)$ of $\epsilon$-hermitian spaces over $A$ is the quotient of the Grothendieck group of $\epsilon$-hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of $W(A)$.

Consider now the rings $A[t]$ and $A[t, t^{-1}]$, endowed with the involution that fixes $t$ and maps $a \in A$ to $a^\circ$. For the ring $A[t, t^{-1}]$ we introduce a variant $W'(A[t, t^{-1}])$ of the Witt group. We first consider the Grothendieck group $Q$ of $\epsilon$-hermitian spaces over $A[t, t^{-1}]$ which are extended from $A$ as $A[t, t^{-1}]$-modules, and its subgroup $N$ generated by the hyperbolic spaces $H(P)$ where $P$ is extended from $A$. We then define $W'(A[t, t^{-1}])$ as $Q/N$. Clearly $W'(A[t, t^{-1}])$ maps canonically to $W(A[t, t^{-1}])$. Here are our results.

**A** (Theorem 3.1). *Let $A$ be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \to W(A[t])$$

*is an isomorphism.*

**B** (THEOREM 5.1). *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi\colon W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

*mapping* $(\xi,\eta)$ *to* $\xi + t\eta$ *is an isomorphism.*

**C** (THEOREM 7.1). *Let A be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi\colon W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$$

*be the canonical homomorphism.*

(a) *If* $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$, *then* $\varphi$ *is surjective.*

(b) *If* $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$, *then* $\varphi$ *is an isomorphism.*

Two examples will be constructed in § 8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

**D** (PROPOSITION 6.8). *Let A be a commutative semilocal ring in which 2 is invertible. Let* $(P,\alpha)$ *be a quadratic space over A. If* $(P,\alpha)$ *is isometric to* $(P, t\cdot\alpha)$ *over* $A[t,t^{-1}]$, *then* $(P,\alpha)$ *is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$Res\colon W(A[t,t^{-1}]) \to W(A)$$

satisfying the following two properties:

- For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(\xi) = 0$.
- For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(t\cdot\xi) = \xi$.

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi \circ \psi\colon W(A) \oplus W(A) \to W(A[t,t^{-1}]),$$

which may fail, as in Example 8.1. However, there exists a residue map $Res\colon W'(A[t,t^{-1}]) \to W(A)$ (Proposition 5.2) which yields the injectivity of $\psi$.

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let $(P, \alpha)$ be any space. Then:*

1. *The space $(P, \alpha) \perp (P, -\alpha)$ is hyperbolic.*
2. *If $L$ is a lagrangian of $(P, \alpha)$, then $(P, \alpha)$ is isometric to $H(L)$.*
3. *If $M$ is a sublagrangian of $(P, \alpha)$, then the map $\alpha$ induces on $M^{\perp}/M$ a natural structure of hermitian space that makes it Witt equivalent to $(P, \alpha)$.*

## 2. K-THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring $A$ we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right $A$-modules and by $K_1(A)$ the abelianized general linear group of $A$: $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over $A$.

For any functor $F$ from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t = 0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1} = 0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t, t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor $LK_1$ turns out to be naturally isomorphic to $K_0$, hence we will denote $LK_i$ by $K_{i-1}$ for $i = 1$ and also for $i = 0$.

THEOREM 2.1. *Let $A$ be any associative ring.*
(a) *For $i = 0$ or $1$ there exists a natural embedding*

$$\lambda_i \colon K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

*such that the composite*

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \longrightarrow LK_i(A) = K_{i-1}(A)$$

*is the identity.*