Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ-hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N - 1$ and after $N - 1$ such transformations we get a linear matrix. $\square$

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1 + \nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1 + \nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1 + \nu t)\tau = \alpha_0\tau(1 + \nu t)\tau = \alpha_0.$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. $\square$

4. THE WITT GROUP OF TORSION MODULES

Let M be a finitely generated right $A[t]$-module and suppose that it is a t-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by $M^\sharp$ the left $A[t]$-module $\mathrm{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A-module, the quotient $A[t, t^{-1}]/A[t]$ can be written as a direct sum

$$A[t, t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots.$$

Thus, to any $f \in \mathrm{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1} : M \to A$, which is defined as the composite of f with the projection onto At^{-1}.

PROPOSITION 4.1. *The map*

$$\partial = \partial_M \colon M^\sharp = \mathrm{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \longrightarrow \mathrm{Hom}_A(M, A) = M^*$$

obtained by associating f_{-1} to f is a functorial A-linear isomorphism.
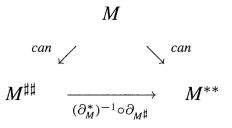
Proof. It is clear that ∂ is A-linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define $\widetilde{g}$ by the (finite !) sum

$$\widetilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \cdots .$$

It is easy to check that $\widetilde{g} \in M^\sharp$, $(\widetilde{g})_{-1} = g$ and $\widetilde{f_{-1}} = f$. Functoriality is clear. $\square$

COROLLARY 4.2. *For any finitely generated t-torsion module M which is projective as an A-module the canonical homomorphism $M \to M^{\sharp\sharp}$ is an isomorphism.*

Proof. It suffices to remark that the diagram

$$
\begin{array}{ccc}
 & M & \\
{}^{can}\swarrow & & \searrow{}^{can} \\
M^{\sharp\sharp} \xrightarrow[\ (\partial^*_M)^{-1}\circ\partial_{M^\sharp}\ ]{} & & M^{**}
\end{array}
$$

commutes and that $M \xrightarrow{\ can\ } M^{**}$ is an isomorphism. $\square$

An ϵ-*hermitian t-torsion space* (or, briefly, a t-*torsion space*) is a pair $(M, \langle,\rangle)$ consisting of a finitely generated t-torsion right $A[t]$-module M which is projective as an A-module, and a perfect ϵ-hermitian pairing $\langle \ , \ \rangle \colon M \times M \to A[t, t^{-1}]/A[t]$. Giving $\langle,\rangle$ is the same, of course, as giving its adjoint $\varphi \colon M \to M^\sharp$ defined by $\varphi(a)(b) = \langle a, b\rangle$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

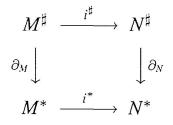$$X^\perp = \{y \in M \mid \langle x, y\rangle = 0 \quad \forall x \in X\} .$$

A *sublagrangian* of (M, φ) is an $A[t]$-submodule L of M which satisfies the following two conditions:

(1) It is contained in its own orthogonal: $L \subseteq L^\perp$.

(2) The quotient M/L is projective over A (which is the same as saying that L, as an A-module, is a direct factor of M).

A sublagrangian L is a *lagrangian* if $L = L^\perp$. A t-torsion space is *metabolic* if it has a lagrangian. The Witt group of t-torsion spaces is the quotient of the Grothendieck group of t-torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

LEMMA 4.3. *Let M and N be finitely generated t-torsion modules and $i \colon N \to M$ an $A[t]$-linear homomorphism. Assume that as A-modules M and N are projective. Then the map $i^\sharp \colon M^\sharp \to N^\sharp$ is surjective (respectively injective) if and only if $i^* \colon M^* \to N^*$ is surjective (respectively injective).*

Proof. Look:

$$
\begin{array}{ccc}
M^\sharp & \xrightarrow{\ i^\sharp\ } & N^\sharp \\
\partial_M \downarrow & & \downarrow \partial_N \\
M^* & \xrightarrow{\ i^*\ } & N^*
\end{array}
$$

$\square$

PROPOSITION 4.4. *Let (M, φ) be a t-torsion space and L an $A[t]$-submodule of M. If M/L is projective over A, then $L = L^{\perp\perp}$ and $L^\perp$ is a direct factor of M as an A-module.*

Proof. First observe that as an A-module L is finitely generated and projective. Let $i \colon L \to M$ be the natural injection. By Lemma 4.3 the map $i^\sharp \circ \varphi$ is surjective, thus the sequence

$$ 0 \longrightarrow L^\perp \xrightarrow{\ j\ } M \xrightarrow{\ i^\sharp \circ \varphi\ } L^\sharp \longrightarrow 0 $$

is exact. Hence $L^\perp$ is a direct factor of M as an A-module; in particular it is A-projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$ 0 \longrightarrow L \xrightarrow{\ i\ } M \xrightarrow{\ j^\sharp \circ \varphi^\sharp\ } (L^\perp)^\sharp \longrightarrow 0. $$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$ 0 \longrightarrow L^{\perp\perp} \longrightarrow M \xrightarrow{\ j^\sharp \circ \varphi\ } (L^\perp)^\sharp \longrightarrow 0 $$

is exact because $L^\perp$ is a direct factor of M as an A-module. Since $\varphi^\sharp = \pm\varphi$, comparing the last two sequences we get the result. $\square$

We now prove a fundamental result on the equivalence of t-torsion spaces.

THEOREM 4.5. *Let (M, φ) be an ϵ-hermitian t-torsion space and L a sublagrangian of (M, φ). The quotient $L^{\perp}/L$ carries a natural structure of t-torsion ϵ-hermitian space and its class in $W_{tors}(A[t])$ is the same as that of (M, φ).*

Proof. We first prove the following lemma.

LEMMA 4.6. *Let (M, φ) be any ϵ-hermitian t-torsion space. The space $(M, \varphi) \perp (M, -\varphi)$ is metabolic.*

Proof of Lemma 4.6. We show that the image $L = \Delta(M)$ of the diagonal map $M \xrightarrow{\Delta} M \oplus M$ is a lagrangian. The condition $L \subseteq L^{\perp}$ is immediately verified. The quotient $(M \oplus M)/L$ is isomorphic to M, hence it is projective over A. It remains to see that $L^{\perp} \subseteq L$. If $(a, b) \in L^{\perp}$ we have $0 = \langle (a, b), (x, x) \rangle = \langle a - b, x \rangle$ for any $x \in M$. Since the pairing $\langle, \rangle$ is perfect, this implies $a = b$, i.e. $(a, b) \in L$. $\square$

We now prove the theorem. By Proposition 4.4, $L^{\perp}$ is a direct factor of M as an A-module. Since $L \subseteq L^{\perp}$ is also a direct factor of M, the quotient $L^{\perp}/L$ is projective. Denoting by $\bar{a}, \bar{b}$ the classes modulo L of two elements $a, b \in L$, we define the hermitian structure of $L^{\perp}/L$ by $\langle \bar{a}, \bar{b} \rangle = \langle a, b \rangle$. It is clear that $\langle a, b \rangle$ only depends on $\bar{a}$ and $\bar{b}$. We first check that this pairing defines a t-torsion space. It is clearly ϵ-hermitian. The injectivity of the adjoint map $L^{\perp}/L \to (L^{\perp}/L)^{\sharp}$ follows immediately from Proposition 4.4. To show surjectivity consider any $A[t]$-linear map $f \colon L^{\perp} \to A[t, t^{-1}]/A[t]$. Since $L^{\perp}$ is a direct factor of M as an A-module, f, by Lemma 4.3, extends to an $A[t]$-linear map $\tilde{f} \colon M \to A[t, t^{-1}]/A[t]$. Choose an $m \in M$ for which $\tilde{f} = \langle m, \cdot \rangle$. If $\tilde{f}$ vanishes on L, then m is in $L^{\perp}$. This proves that $L^{\perp}/L$ is a t-torsion space.

To show that $L^{\perp}/L$ is equivalent to (M, φ) we check that the image of the diagonal map $\Delta \colon L^{\perp} \to M \oplus L^{\perp}/L$ is a lagrangian of $(M, -\varphi) \perp L^{\perp}/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^{\perp})$ is contained in its own orthogonal. Conversely, if $(a, \bar{b}) \in M \oplus L^{\perp}/L$ is orthogonal to every $(x, \bar{x})$, then $\langle a - b, x \rangle = 0$ for every $x \in L^{\perp}$. This means that $a - b$ is in $L^{\perp\perp}$, which by Proposition 4.4 coincides with L. We thus have $(a, \bar{b}) = (a, \bar{a}) \in \Delta(L^{\perp})$. $\square$

The next proposition connects the Witt group of t-torsion spaces with the Witt group of A.

PROPOSITION 4.7. *The isomorphisms*

$$\partial_M \colon \mathrm{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \to \mathrm{Hom}_A(M, A)$$

induce a surjective homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A).$$

Proof. Associating to any t-torsion space (M, φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic t-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A).$$

To find a preimage (M, φ) of a space (M, α) over A consider M as an $A[t]$-module annihilated by t and replace $\alpha \colon M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. $\square$

5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi \colon W(A) \oplus W(A) \to W'(A[t, t^{-1}])$$

mapping (ξ, η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$Res \colon W'(A[t, t^{-1}]) \to W(A)$$

with the following properties:

R_1 : *For any constant space $\xi \in W(A) \subset W'(A[t, t^{-1}])$, $Res(\xi) = 0$.*
R_2 : *For any constant space $\xi \in W(A) \subset W'(A[t, t^{-1}])$, $Res(t \cdot \xi) = \xi$.*

Proof. See Theorem 6.7. $\square$