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**Autor:** Ojanguren, Manuel / Panin, Ivan  
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PROPOSITION 4.7. *The isomorphisms*

$$\partial_M: \text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \rightarrow \text{Hom}_A(M, A)$$

*induce a surjective homomorphism*

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

*Proof.* Associating to any  $t$ -torsion space  $(M, \varphi)$  the hermitian space  $(M, \partial_M \circ \varphi)$  preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic  $t$ -torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

To find a preimage  $(M, \varphi)$  of a space  $(M, \alpha)$  over  $A$  consider  $M$  as an  $A[t]$ -module annihilated by  $t$  and replace  $\alpha: M \rightarrow M^*$  by  $\varphi = \partial_M^{-1} \circ \alpha$ .  $\square$

## 5. THE WITT GROUP OF EXTENDED SPACES

Let  $W'(A[t, t^{-1}])$  be the group defined in the introduction.

THEOREM 5.1. *Let  $A$  be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi: W(A) \oplus W(A) \rightarrow W'(A[t, t^{-1}])$$

*mapping  $(\xi, \eta)$  to  $\xi + t\eta$  is an isomorphism.*

*Proof.* The injectivity of  $\psi$  is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$\text{Res}: W'(A[t, t^{-1}]) \rightarrow W(A)$$

*with the following properties:*

$R_1$ : *For any constant space  $\xi \in W(A) \subset W'(A[t, t^{-1}])$ ,  $\text{Res}(\xi) = 0$ .*

$R_2$ : *For any constant space  $\xi \in W(A) \subset W'(A[t, t^{-1}])$ ,  $\text{Res}(t \cdot \xi) = \xi$ .*

*Proof.* See Theorem 6.7.  $\square$

Assuming this proposition, suppose that for two elements  $\xi, \eta \in W(A)$  we have  $\xi + t \cdot \eta = 0$ . Then  $0 = \text{Res}(\xi + t \cdot \eta) = \eta$  and hence  $\xi = 0$ .

We now turn to the surjectivity of  $\psi$ . We have to show that every hermitian space  $(P, \alpha)$  over  $A[t, t^{-1}]$  with  $P = P_0[t, t^{-1}]$  is Witt equivalent to a space of the form  $(Q_0[t, t^{-1}], \alpha_0) \perp (Q_1[t, t^{-1}], t\alpha_1)$ . Let  $P_1$  be a projective  $A$ -module such that  $P_0 \oplus P_1 = A^n$  for some  $n$ . Replacing  $(P, \alpha)$  by

$$(P_0[t, t^{-1}], \alpha) \perp (P_0[t, t^{-1}], -\alpha(1)) \perp H(P_1[t, t^{-1}]),$$

we may assume that  $P_0$  is free. Replacing  $\alpha$  by  $t^{2N}\alpha$  with a suitable  $N$ , we may also assume that  $\alpha$  maps  $P_0[t]$  into  $P_0^*[t]$ . By Lemma 3.2 we are reduced to the case where  $\alpha = \alpha_0 + t\alpha_1$  for some  $\epsilon$ -hermitian maps  $\alpha_0, \alpha_1: P_0 \rightarrow P_0^*$ .

LEMMA 5.3. *If, for a constant matrix  $\beta$ ,*

$$\alpha = 1 + (t - 1)\beta \in \text{GL}_n(A[t, t^{-1}]) \cap \text{M}_n(A[t]),$$

*then there exists an  $N$  such that  $(1 - \beta)^N \beta^N = 0$ .*

*Proof.* This is Corollary 2.4 of [2]. For the convenience of the reader we reprove it here.

Writing the inverse of  $\alpha$  as a Laurent polynomial and equating coefficients in the identity

$$1 = \alpha\alpha^{-1} = (1 - \beta + t\beta)(\gamma_{-q}t^{-q} + \cdots + \gamma_{-1}t^{-1} + \gamma_0 + \gamma_1t + \cdots + \gamma_pt^p)$$

we get

$$\begin{aligned} (1 - \beta)\gamma_{-q} &= 0, \quad (1 - \beta)\gamma_{-q+1} + \beta\gamma_{-q} = 0, \quad \dots, \\ (1 - \beta)\gamma_{-1} + \beta\gamma_{-2} &= 0, \quad (1 - \beta)\gamma_0 + \beta\gamma_{-1} = 1 \end{aligned}$$

and

$$(1 - \beta)\gamma_1 + \beta\gamma_0 = 0, \quad \dots, \quad (1 - \beta)\gamma_p + \beta\gamma_{p-1} = 0, \quad \beta\gamma_p = 0.$$

From the first line we get  $(1 - \beta)^q \gamma_{-1} = 0$ , from the third  $\beta^{p+1} \gamma_0 = 0$  and then from the middle one  $\beta^{p+1} (1 - \beta)^q = 0$ .  $\square$

We put  $\beta = \alpha(1)^{-1} \alpha_1: P_0 \rightarrow P_0$ , so that

$$\alpha(1)^{-1} \alpha = 1 + (t - 1)\beta.$$

We will repeatedly use the fact that  $\beta$  is adjoint with respect to  $\alpha, \alpha(1), \alpha_0, \alpha_1$ , by which we mean that  $\alpha\beta = \beta^* \alpha$ , and so on. The same clearly holds for any polynomial in  $\beta$  with integral coefficients.

By Lemma 5.3 we can find an integer  $N$  such that  $\beta^N(1 - \beta)^N = 0$ . Denoting by  $\mathbf{Z}[\beta]$  the subring of  $\text{End}_A(P_0)$  generated by  $\beta$  we can write  $\mathbf{Z}[\beta] = \mathbf{Z}[\beta]e \times \mathbf{Z}[\beta](1 - e)$ , where  $e$  is an idempotent of the form  $\beta + \nu$  and  $\nu$  is a nilpotent matrix. Note that  $e$  and  $\nu$  are polynomials in  $\beta$  and therefore they commute with  $\beta$  and with each other. If we decompose  $P_0$  as  $eP_0 + (1 - e)P_0$  and represent  $A$ -linear endomorphisms of  $P_0$  as  $2 \times 2$  block matrices, we have

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 + \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$$

and

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \epsilon\alpha_{12}^* & \alpha_{22} \end{pmatrix} (1 + (t - 1)\beta).$$

Computing the product we see that the condition  $\alpha^* = \epsilon\alpha$  implies that

$$\alpha_{12}(1 - \nu_2) = -\nu_1^*\alpha_{12}, \quad \alpha_{11}^* = \epsilon\alpha_{11} \quad \text{and} \quad \alpha_{22}^* = \epsilon\alpha_{22}.$$

From this we immediately deduce

$$\alpha_{12}(1 - \nu_2)^k = (-\nu_1^*)^k\alpha_{12}$$

for any natural integer  $k$ . Since  $\nu_1$  and  $\nu_2$  are nilpotent, this implies that  $\alpha_{12} = 0$ . Thus  $\alpha$  is of the form

$$\begin{pmatrix} \alpha_{11}t(1 + \nu_1) - \alpha_{11}\nu_1 & 0 \\ 0 & \alpha_{22}(1 + (t - 1)\nu_2) \end{pmatrix}$$

and  $(P_0[t, t^{-1}], \alpha)$  splits as a hermitian space.

Since  $\alpha$ ,  $\alpha_{11}$  and  $\alpha_{22}$  are symmetric, evaluating the above matrix at  $t = 1$  we see that

$$\alpha_{11}\nu_1 = \nu_1^*\alpha_{11} \quad \text{and} \quad \alpha_{22}\nu_1 = \nu_2^*\alpha_{22}.$$

The first block can be written as

$$\sigma_1 = \alpha_{11}t(1 + \nu_1 - t^{-1}\nu_1) = \alpha_{11}t(1 + (1 - t^{-1})\nu_1).$$

Since  $(1 - t^{-1})\nu_1$  is nilpotent, the formal power series

$$\tau_1 = (1 + (1 - t^{-1})\nu_1)^{-1/2} = \sum \binom{-1/2}{k} ((1 - t^{-1})\nu_1)^k$$

is a Laurent polynomial and we can replace the first block by  $\tau_1^*\sigma_1\tau_1 = \alpha_{11}t$ . Similarly, the power series

$$\tau_2 = (1 + (t - 1)\nu_2)^{-1/2} = \sum \binom{-1/2}{k} ((t - 1)\nu_2)^k$$

is a Laurent polynomial and we can replace the second block by  $\tau_2^*\sigma_2\tau_2 = \alpha_{22}$ .

This shows that

$$(P_0[t, t^{-1}], \alpha) \simeq (P_0e[t, t^{-1}], t\alpha_{11}) \perp (P_0(1 - e)[t, t^{-1}], \alpha_{22}),$$

thus proving the surjectivity of  $\psi$ .  $\square$