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Anhang: Appendix. A bijection between hives and Littlewood-Richardson skew tableaux (by William FULTON)

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Appendix. A bijection between hives and Littlewood-Richardson skew tableaux
(by William Fulton)

The aim of this appendix is to give a simple and direct bijection between the hives with given boundary (given by a triple of partitions), and the set of Littlewood-Richardson skew tableaux for the given triple. In principle one could construct such a mapping from [4], but it is simpler to do it directly from hives; in the description we give here, it is easy to see that the map is a bijection, without knowing that the two sets have the same cardinality. As in [4], we produce contratableaux, but there is a standard bijection between these and the original Littlewood-Richardson skew tableaux.

Consider an integral hive, with sides having $n+1$ entries, corresponding to partitions λ, μ, and ν, with $|\nu| = |\lambda| + |\mu|$. The differences down the northwest to southeast border give the partition λ, the differences across the bottom border from right to left give μ, and the differences down the northeast to southwest border give ν (see Theorem 1). The main idea for constructing a skew tableau with a reverse-lattice word is to use the other northwest to southeast rows of entries to construct a chain of subpartitions of λ.

The entries of the hive will be denoted a^i_k, with $1 \le i \le n+1$ and $0 \le k \le n+1-i$. Here the superscript denotes the northwest to southeast row of the entry, with the first row being the long row on the boundary, and the others in order below that; the subscripts number the entries along the rows, from northwest to southeast.

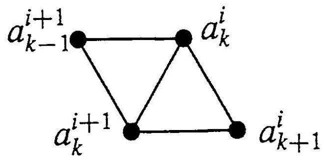
$$
\begin{array}{ccccccc}
 & & & a^1_0 & & & \\
 & & a^2_0 & & a^1_1 & & \\
 & a^3_0 & & a^2_1 & & a^1_2 & \\
a^4_0 & & a^3_1 & & a^2_2 & & a^1_3
\end{array}
$$

Note that $a^1_0 = 0$, and that $\lambda_k = a^1_k - a^1_{k-1}$ for $1 \le k \le n$.

For $1 \le i \le n$ define a sequence $\lambda^{(i)} = (\lambda^{(i)}_1, \dots, \lambda^{(i)}_{n+1-i})$ by setting $\lambda^{(i)}_k = a^i_k - a^i_{k-1}$. Note that $\lambda^{(1)} = \lambda$.

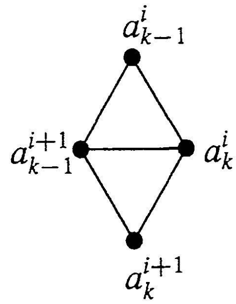
There are three types of rhombus inequalities, depending on the orientation of the rhombus. We first consider two of them:

(1)



This says that $\lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$.

(2)



This says that $\lambda_k^{(i)} \geq \lambda_k^{(i+1)}$.

Together, (1) and (2) say that $\lambda_k^{(i)} \geq \lambda_k^{(i+1)} \geq \lambda_{k+1}^{(i)}$. In particular, each sequence $\lambda^{(i)}$ is weakly decreasing, and we have a nested sequence of partitions: $\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)} \supset \lambda^{(n+1)} = \varnothing$.

For example, the hive

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 10 & & 6 & & & \\
 & & 17 & & 14 & & 10 & & \\
 & 24 & & 21 & & 18 & & 14 & \\
28 & & 26 & & 23 & & 19 & & 15
\end{array}
$$

gives the chain of partitions $(6,4,4,1) \supset (4,4,1) \supset (4,2) \supset (2) \supset \varnothing$.

We identify partitions with Young diagrams, but rotated by 180 degrees, so the diagram for a partition λ has λ_k boxes in the k^{th} row from the bottom, and the rows are lined up on the right. Fill the boxes by putting the integer i in each box of $\lambda^{(i)} - \lambda^{(i+1)}$. The conditions (1) and (2) say exactly that the result T is a skew tableau on this shape, that is, it is weakly increasing across rows and strictly increasing down columns. Such a T is often called a contratableau of shape λ. In our example, T is

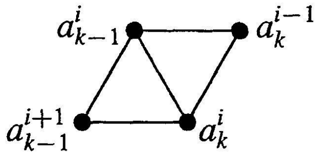
					1
		1	1	1	2
		2	2	3	3
1	1	3	3	4	4

The word $w(T)$ is obtained by reading from left to right in rows, from bottom to top. In the example, $w(T) = 1\,1\,3\,3\,4\,4\,2\,2\,3\,3\,1\,1\,1\,2\,1$.

Let $U(\mu)$ be the tableau of shape μ whose i^{th} row has μ_i entries, all equal to i. The word $w(U(\mu))$ is similarly read from left to right, bottom to top. In our example, $\mu = (4,4,3,2)$, and $w(U(\mu)) = 4\,4\,3\,3\,3\,2\,2\,2\,2\,1\,1\,1\,1$.

Now we consider the last rhombus inequalities:

(3)



These say that $a^{i+1}_{k-1} - a^i_{k-1} \leq a^i_k - a^{i-1}_k$. We claim that this is equivalent to the condition that $w(T) \cdot w(U(\mu))$ is a reverse lattice word [5, §5.2].

This asserts that, if we divide this word at any point, the number of times that i occurs to the right of this point does not exceed the number of times that $i-1$ occurs to the right of this point. We only need to check this at a division corresponding to the place in the k^{th} row from the bottom of T that divides elements strictly smaller than i from elements greater than or equal to i. The number of times that i occurs here is

$$\begin{aligned}
&(\lambda^{(i)}_k - \lambda^{(i+1)}_k) + (\lambda^{(i)}_{k+1} - \lambda^{(i+1)}_{k+1}) + \cdots + (\lambda^{(i)}_{n+1-i} - 0) + \mu_i \\
&\quad = (\lambda^{(i)}_k + \lambda^{(i)}_{k+1} + \cdots + \lambda^{(i)}_{n+1-i}) - (\lambda^{(i+1)}_k + \lambda^{(i+1)}_{k+1} + \cdots + \lambda^{(i+1)}_{n-i}) + \mu_i \\
&\quad = (a^i_{n+1-i} - a^i_{k-1}) - (a^{i+1}_{n-i} - a^{i+1}_{k-1}) + (a^{i+1}_{n-i} - a^i_{n+1-i}) \\
&\quad = a^{i+1}_{k-1} - a^i_{k-1}\,.
\end{aligned}$$

Similarly, the number of times that $i-1$ occurs is

$$(\lambda^{(i-1)}_{k+1} - \lambda^{(i)}_{k+1}) + (\lambda^{(i-1)}_{k+2} - \lambda^{(i)}_{k+2}) + \cdots + (\lambda^{(i-1)}_{n+2-i} - 0) + \mu_{i-1} = a^i_k - a^{i-1}_k\,.$$

Note that the number of times i occurs in all of T is $a^{i+1}_0 - a^i_0 - \mu_i = \nu_i - \mu_i$.

This process is reversible. Given any contratableau T of shape λ such that $w(T) \cdot w(U(\mu))$ is a reverse lattice word, T determines the chain

$\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)} \supset \varnothing$, and from these partitions one successively fills in the entries in the northwest to southeast diagonal rows of the hive; the rhombus inequalities (1)–(3) are automatically satisfied.

To make the story complete, we recall why such contratableaux correspond to Littlewood-Richardson skew tableaux, using standard results about tableaux, as in [5]. However, it may be pointed out that these contratableaux are at least as easy to produce and enumerate as the more classical skew tableaux. First, the condition that $w(T) \cdot w(U(\mu))$ is a reverse lattice word, given that the number of times i occurs in T is $\nu_i - \mu_i$, is equivalent to asserting that $w(T) \cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$ [5, §5.2]. The rectification R of a contratableau T of shape λ is easily seen to be a tableau of shape λ, and with the same property that $w(R) \cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$. The correspondence between tableaux and contratableaux of shape λ is a bijection, by reversing the rectification process.

Now the condition that $w(R) \cdot w(U(\mu))$ be Knuth equivalent to $w(U(\nu))$ is equivalent to the condition that $R \cdot U(\mu) = U(\nu)$ in the plactic monoid of tableaux [5, §2.1]. It is easy to see, from the definition of multiplying tableaux by column bumping entries of the first tableau into the second [5, §A.2], that if R and S are tableaux with $R \cdot S = U(\beta)$, then S must be equal to $U(\alpha)$ for some partition α. This gives a correspondence between the set of tableaux R that we are looking at and the set of pairs (R, S) with R of shape λ, S of shape μ, whose product is the tableau $U(\nu)$. There is a standard construction [5, §5.1] between these pairs and the set of skew tableau on the shape ν/λ of content μ whose word is a reverse-lattice word.

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