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Artikel: ALMOST COMPLEX STRUCTURES ON 8-MANIFOLDS

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Kapitel: 2. Stable almost complex structures

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Theorem 4 in the general case is also due to the first author. The present joint note grew from the second author's observation that the original proof can be simplified and be made more geometric by appealing to considerations in [5].

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2. Stable almost complex structures

To derive Theorem 4(a) from Theorem 3(a) we merely have to show that the condition $w_8(M) \in \mathrm{Sq}^2 H^6(M;\mathbf{Z})$ is void in case I and equivalent to $\chi(M) \equiv 0 \mod 2$ (i.e. $w_8(M) = 0$) in case II.

Indeed, in case I we find an integral lift u of $w_2(M)$ whose free part is indivisible. Then there is a dual element $u' \in H^6(M;\mathbf{Z})$ such that $uu' = 1 \in H^8(M;\mathbf{Z})$. By the Wu formula it follows that

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = w_2(M)\, H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = H^8(M;\mathbf{Z}_2)\,.$$

In case II, on the other hand, we can lift $w_2(M)$ to a torsion class $u \in H^2(M;\mathbf{Z})$, thus

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = 0\,.$$

3. The top-dimensional obstruction

Assume we have an almost complex structure J_0 over M with a disc D^8 removed (which is homotopy equivalent to the 7-skeleton of M). Thinking of an almost complex structure J as a section of the SO_8/U_4-bundle associated to TM, we may interpret J_0 as such a section defined only over the 7-skeleton of M. The obstruction $\mathfrak{o}(M, J_0)$ to extending J_0 to an almost complex structure on M then lives in

$$H^8(M;\pi_7(\mathrm{SO}_8/U_4)) \cong \pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z}\oplus\mathbf{Z}_2\,.$$

See [13] for references to the computations of the homotopy group above and others used below. The homotopy group involved here is in fact the