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Artikel: ALMOST COMPLEX STRUCTURES ON 8-MANIFOLDS
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It follows that

$$\mathfrak{o}_0(X_m \# X_n \# S^4 \times S^4, J_m + J_n + J_0') = 3(m(m+1) + n(n+1))/7$$

(and $\mathfrak{o}_2 = 0$). With the given constraints on m and n, this can only be zero for $m = n = 0$ (even if we allow for the orientation of the summands to be changed).

For the allowed choices of m and n, this connected sum is homotopy equivalent to $\mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$. The fact that the homotopy equivalence $X_m, X_n \simeq X_0$ induces a homotopy equivalence of the connected sums is a simple consequence of the Whitehead theorem, since we are dealing with simply-connected manifolds. This concludes the proof of Proposition 6. □

4. Existence of almost complex structures

In this section we prove Theorem 4(b). We already know that condition (b) (i) is necessary. We now show that condition (b) (ii) is necessary.

Given an almost complex structure J on M, we have

$$2\chi(M) - 2c_1(J)\, c_3(J) + c_2(J)^2 - p_2(M) = 0$$

by (1). In the sequel we suppress M and J. In case II_0, c_1 is a torsion class, so this simplifies to

$$2\chi + c_2^2 - p_2 = 0\,.$$

Squaring the relation $p_1 = c_1^2 - 2c_2$, and again observing that c_1 is a torsion class, we get $p_1^2 = 4c_2^2$. Multiplying the equation above by 4 and substituting p_1^2 for $4c_2^2$ yields condition (b) (ii).

In fact, this argument also shows that (b) (ii) is a sufficient condition. By (a) we have a stable a.c.s. $\widetilde{J}$ on M and thus a corresponding J_0 as in Section 3. If condition (b) (ii) holds, then reversing the argument just given we find

$$4\mathfrak{o}_0(M, J_0) = 2\chi(M) + c_2(J_0)^2 - p_2(M) = 0\,.$$

Since J_0 is induced by $\widetilde{J}$, the stable part $\mathfrak{o}_2$ of the obstruction vanishes as well, so J_0 extends to an almost complex structure on M.

Next we prove that condition (b) (i) is sufficient for the existence of an a.c.s. We begin with a preparatory lemma.

LEMMA 10. *Let M be an 8-manifold with $b_2 > 0$ which satisfies the assumptions of Theorem* 4(a). *Then there is a family $\widetilde{J}_k$, $k \in \mathbf{Z}$, of stable almost complex structures on M such that*

$$c_1(\widetilde{J}_k)\, c_3(\widetilde{J}_k) = c_1(\widetilde{J}_0)c_3(\widetilde{J}_0) + 4k$$

and

$$c_2(\widetilde{J}_k) = c_2(\widetilde{J}_0) \text{ modulo 2-torsion},$$

so in particular $c_2(\widetilde{J}_k)^2 = c_2(\widetilde{J}_0)^2$.

Proof. Case I: By Theorem 3(a) we can find a stable a.c.s. $\widetilde{J}_0$ such that the free part x of $c_1(\widetilde{J}_0)$ is indivisible. By Poincaré duality there is an element $x' \in H^6(M;\mathbf{Z})$ such that $c_1(\widetilde{J}_0)x' = xx' = 1$. Then the pair $(c_1(\widetilde{J}_0), c_3(\widetilde{J}_0) + 4kx')$, $k \in \mathbf{Z}$, still satisfies the assumptions of Theorem 3(a), so there are stable almost complex structures $\widetilde{J}_k$, $k \in \mathbf{Z}$, with

$$c_1(\widetilde{J}_k) = c_1(\widetilde{J}_0) \quad \text{and} \quad c_3(\widetilde{J}_k) = c_3(\widetilde{J}_0) + 4kx'.$$

This is the desired family, since the relation $p_1 = c_1^2 - 2c_2$ shows that c_2 is determined modulo 2-torsion by c_1 and p_1.

Case II_+: Let $\widetilde{J}_0$ be a stable a.c.s. such that $c_1(\widetilde{J}_0)$ is torsion. Let x be an indivisible element of $H^2(M;\mathbf{Z})$ and x' a dual element of $H^6(M;\mathbf{Z})$, i.e. $xx' = 1$ and $yx' = 0$ for all y in a complement of $\mathbf{Z}x \subset H^2(M;\mathbf{Z})$ (of course x' depends on the choice of this complement). Then by Theorem 3(a) there exists a stable a.c.s. $\widetilde{J}_k$ with

$$c_1(\widetilde{J}_k) = c_1(\widetilde{J}_0) + 2x \quad \text{and} \quad c_3(\widetilde{J}_k) = c_3(\widetilde{J}_0) + 2kx'.$$

This family has the desired properties. □

Now, assuming that condition (b)(i) is satisfied, we choose a stable a.c.s. $\widetilde{J}$ on M as in the proof of the preceding lemma. Hence we get an a.c.s. J_1 on $M - D^8$ with

$$\mathfrak{o}(M, J_1) = (a_1, 0) \in \mathbf{Z} \oplus \mathbf{Z}_2.$$

If a_1 is even, then by formulae (1) and (2) for $\mathfrak{o}$ and the preceding lemma, we can find a different a.c.s. J_1' on $M - D^8$ with $\mathfrak{o}(M, J_1') = (0, 0)$, so that J_1' extends to an a.c.s. on M.

We complete the proof of Theorem 4 by showing that a_1 has to be even. Since

$$\mathfrak{o}(M \# \mathbf{H}P^2 \# \mathbf{H}P^2, J_1 + J_0 + J_0) = (a_1 - 2, 0),$$

we can find an 8-manifold M_2 and an a.c.s. J_2 on $M_2 - D^8$ with

$$\mathfrak{o}(M_2, J_2) = (-a_2, 0)$$

with $a_2 > 0$ of the same parity as a_1, and

$$\chi(M_2) - \tau(M_2) = \chi(M) - \tau(M) \equiv 0 \mod 4\,.$$

Then

$$\mathfrak{o}(M_2 \# a_2 S^4 \times S^4, J_2 + a_2 J_0') = (0,0)\,.$$

So $M_2 \# a_2 S^4 \times S^4$ admits an a.c.s. Now compute

$$\begin{aligned}(\chi - \tau)(M_2 \# a_2 S^4 \times S^4) &= \chi(M_2) + 2a_2 - \tau(M_2) \\ &\equiv 2a_2 \mod 4\,.\end{aligned}$$

By the necessity of condition (b)(i) we conclude $a_1 \equiv a_2 \equiv 0 \mod 2$. □

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