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Artikel: GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS
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above analytic (i.e. C^*-algebra) K-theory groups. In all computed examples this map is an isomorphism. The picture that emerges is of two parallel theories: one analytic and one geometric. Elliptic operators provide a map from the geometric to the analytic theory. We give evidence for the conjecture that this map is always an isomorphism. In particular we prove that the map is injective for foliations with negatively curved leaves. We then explore some corollaries of this isomorphism conjecture. The injectivity is related through the work of G. G. Kasparov and A. S. Miscenko to the Novikov higher signature problem. We show how this problem leads to a conjecture on the invariance of certain foliation characteristic classes under leaf-wise homotopy equivalence. The surjectivity is related to a number of well-known C^*-algebra problems, such as the non-existence of idempotents in the reduced C^*-algebra of any torsion-free discrete group.

2. Lie group actions

G denotes a Lie group and X denotes a C^∞-manifold without boundary. Both G and X are assumed to be Hausdorff and second countable. G and X may have countably many connected components. G may be a countable discrete group.

DEFINITION 1. A C^∞ (right) action $X \times G \to X$ of G on X is *proper* if the map $X \times G \to X \times X$ given by

$$(x, g) \mapsto (x, xg)$$

is proper (i.e. the inverse image of any compact set is compact).

TERMINOLOGY. A *G-manifold* is a C^∞-manifold with a given (right) C^∞ G-action. If X, Y are G-manifolds a *G-map* from X to Y is a C^∞ G-equivariant map $f: X \to Y$. A G-manifold X is *proper* if the action of G on X is proper. A subset Δ of a proper G-manifold is *G-compact* if the image of Δ in the quotient space X/G is compact. A *G-vector bundle* on a G-manifold X is a C^∞-vector bundle E on X such that E is itself a G-manifold, the projection $E \to X$ is a G-map, and for each $(x, g) \in X \times G$ the map $E_x \to E_{xg}$ given by

$$u \mapsto ug \qquad (u \in E_x)$$

is linear.

A *G-vector bundle with G-compact support* on a proper G-manifold X is a triple (E_0, E_1, σ) where E_0, E_1 are G-vector bundles on X, $\sigma\colon E_0 \to E_1$ is a G-map which is linear on each fibre and Support(σ) is G-compact, where

$$\text{Support}(\sigma) = \{x \in X \mid \sigma\colon E_{0x} \to E_{1x} \text{ is not an isomorphism}\}.$$

For a G-manifold X, the analytic K-theory is the K-theory of the reduced crossed-product C^*-algebra $C_0(X) \rtimes G$. Here $C_0(X)$ is the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity. We now proceed to define the geometric K-theory, denoted $K^*(X, G)$, and the natural map

$$K^i(X, G) \to K_i[C_0(X) \rtimes G] \qquad (i = 0, 1).$$

In doing this the G-manifold X will be "approximated" by proper G-manifolds. Note that the action of G on X is *not* required to be proper. Of special interest is the case when X is a point. For this case $C_0(\cdot) \rtimes G = C^*(G)$ where $C^*(G)$ is the reduced C^*-algebra of G.

Let Z be a proper G-manifold. $V_G^0(Z)$ denotes the collection of all complex G-vector bundles (E_0, E_1, σ) on Z with G-compact support. A group $K_G^0(Z)$ is defined by imposing on $V_G^0(Z)$ the same equivalence relation used by Atiyah-Segal ([5], [31])

$$K_G^0(Z) = V_G^0(Z)/\sim.$$

Addition in $K_G^0(Z)$ is given by direct sum $\xi \oplus \xi'$ of G-vector bundles with G-compact support.

To define $K_G^1(Z)$ let G act on $Z \times \mathbf{R}$ by:

$$(p, t)\, g = (pg, t)$$

$(p \in Z,\ t \in \mathbf{R},\ g \in G)$. Set $V_G^1(Z) = V_G^0(Z \times \mathbf{R})$. Then

$$K_G^1(Z) = K_G^0(Z \times \mathbf{R}).$$

The basic properties of $K_G^*(Z)$ are stated and proved almost exactly as Atiyah-Segal did for compact G.

THOM ISOMORPHISM THEOREM. *On the proper G-manifold Z let E be an $\mathbf{R}$ G-vector bundle with a given G-invariant* Spin^c*-structure. Then*

$$K_G^*(Z) \cong K_G^*(E).$$

REMARK 2. The group $K_G^*(Z)$ is defined and used *only* for proper G-manifolds Z.

DEFINITION 3. Let X be a G-manifold. A *K-cocycle for* (X, G) is a triple (Z, ξ, f) such that

(1) Z is a proper G-manifold;

(2) $f: Z \to X$ is a G-map;

(3) $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$.

T^*Z is the cotangent bundle of Z and f^*T^*X is the pull-back to Z via f of T^*X. In (Z, ξ, f) all structures are C^∞ and G-equivariant.

The main result of this section is the construction of a canonical map μ from K-cocycles to the K-theory of the reduced crossed-product C^*-algebra $C_0(X) \rtimes G$.

THEOREM 4. *Each K-cocycle for (X, G) canonically determines an element in $K_*[C_0(X) \rtimes G]$.*

Outline of proof. First assume that $f: Z \to X$ is a submersion of Z onto X. Let τ be the cotangent bundle along the fibres of f. Using the Thom isomorphism theorem $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines an element $\eta \in V_G^*(\tau)$. For $x \in X$, set $Z_x = f^{-1}(x)$. Then $\eta \in V_G^*(\tau)$ restricts to give $\eta_x \in V^*(T^*Z_x)$, which is the symbol of an elliptic operator on Z_x. Hence η is the symbol of a G-equivariant family D of elliptic operators, parametrized by the points of X. The K-theory index of D is the desired element of $K_*[C_0(X) \rtimes G]$:

$$\operatorname{Index}(D) \in K_*[C_0(X) \rtimes G]\,.$$

If $f: Z \to X$ is not a submersion, then form the commutative diagram

$$\begin{array}{ccc} & & X \times Z \\ & {}^{i}\nearrow & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & X \end{array}$$

where $i(z) = (f z, z)$ and $p(x, z) = x$. Using the Thom isomorphism theorem, $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines $\xi' \in V_G^*(T^*(X \times Z) \oplus p^*T^*X)$. The desired element of $K_*[C_0(X) \rtimes G]$ is then obtained as above from $(X \times Z, \xi', p)$. □

NOTATION. With D as in the proof of Theorem 4, $\operatorname{Index}(D) \in K_*[C_0(X) \rtimes G]$ will be denoted $\mu(Z, \xi, f)$. Observe that $\mu(Z, \xi, f)$ is the analytic index of the K-cocycle (Z, ξ, f). For $\xi \in V_G^i(T^*Z \oplus f^*T^*X)$, one has

$$\mu(Z, \xi, f) \in K_i[C_0(X) \rtimes G] \qquad (i = 0, 1)\,.$$

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & X & \end{array}$$

where Z_1, Z_2, X are G-manifolds with Z_1, Z_2 proper and f_1, f_2, h are G-maps. Using the Thom isomorphism theorem there is then a Gysin map

$$h_!\colon K^i_G(T^*Z_1 \oplus f_1^* T^*X) \to K^i_G(T^*Z_2 \oplus f_2^* T^*X) \qquad (i = 0, 1)\,.$$

Just as for the ordinary analytic index of an elliptic operator on a G-manifold (with G compact) [6], the main property of the index μ is its invariance with respect to push-forward:

THEOREM 5. *The index map μ is compatible with Gysin maps in the following sense. If $\xi_1 \in V^*_G(T^*Z \oplus f_1^* T^*X)$, then $\mu(Z_1, \xi_1, f_1) = \mu(Z_2, h_!(\xi_1), f_2)$.*

REMARK 6. Theorems 4 and 5 indicate how to define the geometric K-theory $K^*(X, G)$ and the natural map

$$\mu\colon K^i(X, G) \to K_i[C_0(X) \rtimes G]\,.$$

For a G-manifold X, let $\Gamma(X, G)$ be the collection of all K-cocycles (Z, ξ, f) for (X, G). On $\Gamma(X, G)$ impose the equivalence relation $\sim$, where $(Z, \xi, f) \sim (Z', \xi', f')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z'' & \xleftarrow{h'} & Z' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f''} & \swarrow{\scriptstyle f'} & \\ & & X & & \end{array}$$

with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 7. $K^*(X, G) = \Gamma(X, G)/\sim$.

The equivalence relation $\sim$ could also be defined as the equivalence relation generated by three elementary steps:

(1) cobordism;

(2) vector bundle modification;

(3) direct sum – disjoint union.

Addition in $K^*(X,G)$ is given by disjoint union of K-cocycles. Further,

$$K^*(X,G) = K^0(X,G) \oplus K^1(X,G),$$

where $K^i(X,G)$ is the subgroup of $K^*(X,G)$ determined by all K-cocycles (Z,ξ,f) with $\xi \in V^i_G(T^*Z \oplus f^*T^*X)$. The natural homomorphism of abelian groups

$$K^i(X,G) \to K_i[C_0(X) \rtimes G]$$

is defined by

$$(Z,\xi,f) \mapsto \mu(Z,\xi,f).$$

CONJECTURE. *For any G-manifold X, $\mu\colon K^i(X,G) \to K_i[C_0(X) \rtimes G]$ is an isomorphism.*

This conjecture is known to be true if X is a proper G-manifold. If X is proper there is a commutative diagram

$$\begin{array}{ccc} K^*(X,G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ {\scriptstyle i_t}\searrow & & \swarrow{\scriptstyle \alpha} \\ & K^*_G(X) & \end{array}$$

in which each arrow is an isomorphism. $i_t\colon K^*(X,G) \to K^*_G(X)$ maps a K-cocycle (Z,ξ,f) to its topological index, and $\alpha \circ \mu\colon K^*(X,G) \to K^*_G(X)$ maps a K-cocycle (Z,ξ,f) to its analytic index. If G is compact then any G-manifold is proper and commutativity of the diagram is equivalent to the Atiyah-Singer index theorems of [6], [7], [8].

3. HOMOTOPY QUOTIENT

Let W be a topological space. $V^0(W)$ denotes the collection of all complex vector bundles (E_0, E_1, σ) on W with compact support. Thus E_0, E_1 are complex vector bundles on W and $\sigma\colon E_0 \to E_1$ is a morphism of complex vector bundles with Support(σ) compact, where

$$\text{Support}(\sigma) = \{p \in W \mid \sigma\colon E_{0p} \to E_{1p} \text{ is not an isomorphism}\}.$$

Also $V^1(W) = V^0(W \times \mathbf{R})$.

Suppose given an $\mathbf{R}$-vector bundle F on W. Following [9], a *twisted by F K-cycle* on W is a triple (M,ξ,ϕ) such that: