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scalar product $(T, V) = \operatorname{Re} \sum \bar{T}_i V_i$ on \mathfrak{p} we have $(T, V\lambda) = (T\bar{\lambda}, V)$, $\lambda \in \mathbf{F}$, therefore \mathfrak{s}^\perp is a \mathbf{F} -subspace of \mathfrak{p} .

An element k of $K \cap H$ is characterized by $k \in K$ and $k \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$, i.e. $k \cdot \mathfrak{s} = \mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of \mathfrak{p} and \mathfrak{s} as \mathbf{F} -vector spaces. Taking a \mathbf{F} -basis of \mathfrak{p} according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, it follows that

$$K = U(1; \mathbf{F}) \times U(n'; \mathbf{F}), \quad K \cap H = U(1; \mathbf{F}) \times U(d'; \mathbf{F}) \times U(n' - d'; \mathbf{F}).$$

But $U(n' - d'; \mathbf{F})$ acts transitively on the unit sphere of $\mathbf{F}^{n' - d'}$, which implies our claim.

If $T, T' \in \mathfrak{s}^\perp$ are two unit vectors, there exists $k_o \in K \cap H$ such that $k_o \cdot T = T'$. Thus

$$\begin{aligned} R_{\exp tT'}^* v(gK) &= \int_K v(gkk_o \exp(tT)k_o^{-1}H) dk \\ &= \int_K v(gk \exp(tT)H) dk = R_{\exp tT}^* v(gK). \end{aligned}$$

In particular $R_{\exp tT}^* v$ is an even function of t .

Going back to (23), we now take as (X_j) an orthonormal \mathbf{R} -basis of \mathfrak{p} according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. The $n - d$ basis vectors in \mathfrak{s}^\perp give the same contribution to the right hand side, whereas the d vectors in \mathfrak{s} generate one parameters subgroups of H and give no contribution; indeed $\exp tV \cdot \operatorname{Exp} \mathfrak{s} = \operatorname{Exp} \mathfrak{s}$ for $V \in \mathfrak{s}$, since \mathfrak{s} is a Lie triple system by Section 4.3 c. This completes the proof. \square

6.5 MULTITEMPORAL WAVES

We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X = G/K$, $Y = G/H$. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ be the respective Lie algebras, and \mathfrak{t} a vector subspace of \mathfrak{g} such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let K_1, \dots, K_p be a basis of \mathfrak{k} , complemented by $H_1, \dots, H_q \in \mathfrak{h}$ so that the K_i 's and H_j 's are a basis of $\mathfrak{k} + \mathfrak{h}$, and let T_1, \dots, T_r be a basis of \mathfrak{t} . We shall use the same notations for the corresponding left-invariant vector fields on G , e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with $f \in C^\infty(G)$, $g \in G$, $s \in \mathbf{R}$. We denote by $\mathbf{D}(G)$ the algebra of all left invariant differential operators on G , by $\mathbf{D}(G)^K$ the subalgebra of right

K -invariant operators and by $\mathbf{D}(X)$ the algebra of G -invariant differential operators on X . For $s = (s_1, \dots, s_r) \in \mathbf{R}^r$, let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r.$$

We recall that, for $g, t \in G$,

$$R_t^* v(gK) = \int_K v(gktH) dk.$$

THEOREM 17. *Let G be a Lie group, H, K Lie subgroups, with K compact and $X = G/K$, $Y = G/H$.*

(i) *For any $P \in \mathbf{D}(X)$ there exists $Q(\partial)$, a constant coefficients differential operator on \mathbf{R}^r , with $\text{order}(Q) \leq \text{order}(P)$, such that for any $v \in C^\infty(Y)$, $x \in X$,*

$$(25) \quad PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x) \Big|_{s=0}.$$

(ii) *Assume furthermore that \mathfrak{t} is a Lie subalgebra of \mathfrak{g} with $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$, and let T denote the connected Lie subgroup of G with Lie algebra \mathfrak{t} . Then for any $P \in \mathbf{D}(X)$ there exists a right-invariant differential operator Q on T , with $\text{order}(Q) \leq \text{order}(P)$, such that*

$$(26) \quad P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x)$$

for $v \in C^\infty(Y)$; here $P_{(x)}$ acts on the variable $x \in X$ and $Q_{(t)}$ acts on $t \in T$.

Thus $R_t^* v(x)$, as a function of $(x, t) \in X \times T$, solves the generalized “multitemporal” wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables $(x, s) \in X \times \mathbf{R}^r$ at the time $s = 0$.

Proof. In order to work on G rather than on its homogeneous spaces, we define $w(g) = v(gH)$ and, for $g, t \in G$,

$$(27) \quad F(g, t) = (R_t^* v)(gK) = \int_K w(gkt) dk,$$

so that $F(gk, k'th) = F(g, t)$ for any $k, k' \in K$, $h \in H$, and

$$F(g, e) = (R^* v)(gK) = \int_K w(gk) dk.$$

Let $P \in \mathbf{D}(X)$ be given. Since K is compact the coset space $X = G/K$ is reductive and there exists $D \in \mathbf{D}(G)^K$ such that ([9], p. 285)

$$(28) \quad (Pf)(gK) = D_{(g)}(f(gK))$$

for $f \in C^\infty(X)$, $g \in G$.

To transfer derivatives from g to t we observe that, by the invariance of D under left translation by gk and right translation by k ,

$$D_{(g)}w(gkt) = D_{(x)}w(gkxt)|_{x=e},$$

where g, x, t are variables in G . Integrating over K it follows that

$$(29) \quad D_{(g)}F(g, t) = D_{(x)}F(g, xt)|_{x=e},$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \dots K_p^{\beta_p} T_1^{\alpha_1} \dots T_r^{\alpha_r} H_1^{\gamma_1} \dots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of $\mathbf{D}(G)$. Setting apart the terms with $\beta = \gamma = 0$, we can thus write, for some $E_i, F_j \in \mathbf{D}(G)$ and some constant coefficients a_α ,

$$(30) \quad D = D' + \sum_{i=1}^p K_i E_i + \sum_{j=1}^q F_j H_j, \quad D' = \sum_{\alpha} a_{\alpha_1 \dots \alpha_r} T_1^{\alpha_1} \dots T_r^{\alpha_r}.$$

If we replace $D_{(x)}$ by (30) in (29), the second term $(K_i E_i)_{(x)} F(g, xt)|_{x=e}$ vanishes because $K_i \in \mathfrak{k}$ and $F(g, kxt) = F(g, t)$. In the third term the left invariant vector field $H_j \in \mathfrak{h}$ acts by

$$(H_j)_{(x)} F(g, xt) = \partial_s F(g, x \exp(sH_j)t)|_{s=0},$$

and this vanishes too whenever t normalizes H , because $F(g, xth) = F(g, xt)$.

Since $t = e$ in case (i), or $t \in T$ with $Ht = tH$ in case (ii), we finally obtain for both cases (in multi-index notation)

$$(31) \quad \begin{aligned} D_{(g)}F(g, t) &= D'_{(x)}F(g, xt)|_{x=e} \\ &= \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} F(g, (\exp s_1 T_1 \dots \exp s_r T_r)t)|_{s=0} \\ &= \left(\sum_{\alpha} a_{\alpha} \partial_s^{\alpha} \right) F(g, t(s)t)|_{s=0}. \end{aligned}$$

Let the operator Q be defined by

$$Qf(t) = \sum_{\alpha} a_{\alpha} \partial_s^{\alpha} f(t(s)t)|_{s=0},$$

a right invariant differential operator on the group T in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii). \square