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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

Autor: Rouvière, François

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scalar product $(T,V) = \mathrm{Re}\sum \overline{T}_i V_i$ on $\mathfrak{p}$ we have $(T, V\lambda) = (T\overline{\lambda}, V)$, $\lambda \in \mathbf{F}$, therefore $\mathfrak{s}^\perp$ is a $\mathbf{F}$-subspace of $\mathfrak{p}$.

An element k of $K \cap H$ is characterized by $k \in K$ and $k \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$, i.e. $k \cdot \mathfrak{s} = \mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of $\mathfrak{p}$ and $\mathfrak{s}$ as $\mathbf{F}$-vector spaces. Taking a $\mathbf{F}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$, it follows that

$$K = U(1;\mathbf{F}) \times U(n';\mathbf{F}), \quad K \cap H = U(1;\mathbf{F}) \times U(d';\mathbf{F}) \times U(n'-d';\mathbf{F}).$$

But $U(n'-d';\mathbf{F})$ acts transitively on the unit sphere of $\mathbf{F}^{n'-d'}$, which implies our claim.

If $T, T' \in \mathfrak{s}^\perp$ are two unit vectors, there exists $k_o \in K \cap H$ such that $k_o \cdot T = T'$. Thus

$$\begin{aligned} R^*_{\exp tT'} v(gK) &= \int_K v\left(gkk_o \exp(tT) k_o^{-1} H\right) dk \\ &= \int_K v\left(gk \exp(tT) H\right) dk = R^*_{\exp tT} v(gK). \end{aligned}$$

In particular $R^*_{\exp tT} v$ is an even function of t.

Going back to (23), we now take as (X_j) an orthonormal $\mathbf{R}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}^\perp$. The $n-d$ basis vectors in $\mathfrak{s}^\perp$ give the same contribution to the right hand side, whereas the d vectors in $\mathfrak{s}$ generate one parameters subgroups of H and give no contribution; indeed $\exp tV \cdot \mathrm{Exp}\, \mathfrak{s} = \mathrm{Exp}\, \mathfrak{s}$ for $V \in \mathfrak{s}$, since $\mathfrak{s}$ is a Lie triple system by Section 4.3 **c**. This completes the proof. □

6.5 Multitemporal waves

We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X = G/K$, $Y = G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the respective Lie algebras, and $\mathfrak{t}$ a vector subspace of $\mathfrak{g}$ such that

$$\mathfrak{g} = (\mathfrak{k} + \mathfrak{h}) \oplus \mathfrak{t}.$$

Let $K_1, \ldots, K_p$ be a basis of $\mathfrak{k}$, complemented by $H_1, \ldots, H_q \in \mathfrak{h}$ so that the K_i's and H_j's are a basis of $\mathfrak{k} + \mathfrak{h}$, and let $T_1, \ldots, T_r$ be a basis of $\mathfrak{t}$. We shall use the same notations for the corresponding left-invariant vector fields on G, e.g.

$$K_i f(g) = \partial_s f(g \exp sK_i)|_{s=0},$$

with $f \in C^\infty(G)$, $g \in G$, $s \in \mathbf{R}$. We denote by $\mathbf{D}(G)$ the algebra of all left invariant differential operators on G, by $\mathbf{D}(G)^K$ the subalgebra of right

K-invariant operators and by $\mathbf{D}(X)$ the algebra of G-invariant differential operators on X. For $s = (s_1, \dots, s_r) \in \mathbf{R}^r$, let

$$t(s) = \exp s_1 T_1 \cdots \exp s_r T_r \, .$$

We recall that, for $g, t \in G$,

$$R_t^* v(gK) = \int_K v(gktH)\, dk \, .$$

THEOREM 17. *Let G be a Lie group, H, K Lie subgroups, with K compact and $X = G/K$, $Y = G/H$.*

(i) *For any $P \in \mathbf{D}(X)$ there exists $Q(\partial)$, a constant coefficients differential operator on $\mathbf{R}^r$, with order(Q) $\leq$ order(P), such that for any $v \in C^\infty(Y)$, $x \in X$,*

$$PR^* v(x) = Q(\partial_s) R_{t(s)}^* v(x)\big|_{s=0} \, . \tag{25}$$

(ii) *Assume furthermore that $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{h}$, and let T denote the connected Lie subgroup of G with Lie algebra $\mathfrak{t}$. Then for any $P \in \mathbf{D}(X)$ there exists a right-invariant differential operator Q on T, with order(Q) $\leq$ order(P), such that*

$$P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x) \tag{26}$$

for $v \in C^\infty(Y)$; here $P_{(x)}$ acts on the variable $x \in X$ and $Q_{(t)}$ acts on $t \in T$.

Thus $R_t^* v(x)$, as a function of $(x, t) \in X \times T$, solves the generalized "*multitemporal*" wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables $(x, s) \in X \times \mathbf{R}^r$ at the time $s = 0$.

Proof. In order to work on G rather than on its homogeneous spaces, we define $w(g) = v(gH)$ and, for $g, t \in G$,

$$F(g, t) = \left(R_t^* v\right)(gK) = \int_K w(gkt)\, dk \, , \tag{27}$$

so that $F(gk, k'th) = F(g, t)$ for any $k, k' \in K$, $h \in H$, and

$$F(g, e) = \left(R^* v\right)(gK) = \int_K w(gk)\, dk \, .$$

Let $P \in \mathbf{D}(X)$ be given. Since K is compact the coset space $X = G/K$ is reductive and there exists $D \in \mathbf{D}(G)^K$ such that ([9], p. 285)

$$(Pf)(gK) = D_{(g)}(f(gK)) \tag{28}$$

for $f \in C^\infty(X)$, $g \in G$.

To transfer derivatives from g to t we observe that, by the invariance of D under left translation by gk and right translation by k,

$$D_{(g)}w(gkt) = D_{(x)}w(gkxt)\big|_{x=e} ,$$

where g, x, t are variables in G. Integrating over K it follows that

$$D_{(g)}F(g,t) = D_{(x)}F(g,xt)\big|_{x=e} , \tag{29}$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \cdots K_p^{\beta_p} T_1^{\alpha_1} \cdots T_r^{\alpha_r} H_1^{\gamma_1} \cdots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of $\mathbf{D}(G)$. Setting apart the terms with $\beta = \gamma = 0$, we can thus write, for some $E_i, F_j \in \mathbf{D}(G)$ and some constant coefficients a_α,

$$D = D' + \sum_{i=1}^{p} K_i E_i + \sum_{j=1}^{q} F_j H_j , \quad D' = \sum_\alpha a_{\alpha_1 \dots \alpha_r} T_1^{\alpha_1} \cdots T_r^{\alpha_r} . \tag{30}$$

If we replace $D_{(x)}$ by (30) in (29), the second term $(K_i E_i)_{(x)} F(g,xt)\big|_{x=e}$ vanishes because $K_i \in \mathfrak{k}$ and $F(g,kxt) = F(g,t)$. In the third term the left invariant vector field $H_j \in\in \mathfrak{h}$ acts by

$$(H_j)_{(x)} F(g,xt) = \partial_s F(g, x \exp\left(sH_j\right) t)\big|_{s=0} ,$$

and this vanishes too whenever t normalizes H, because $F(g,xth) = F(g,xt)$.

Since $t = e$ in case (i), or $t \in T$ with $Ht = tH$ in case (ii), we finally obtain for both cases (in multi-index notation)

$$\begin{aligned} D_{(g)}F(g,t) &= D'_{(x)}F(g,xt)\big|_{x=e} \\ &= \sum_\alpha a_\alpha \partial_s^\alpha F(g, (\exp s_1 T_1 \cdots \exp s_r T_r)\, t)\Big|_{s=0} \\ &= \left(\sum_\alpha a_\alpha \partial_s^\alpha \right) F(g, t(s)t)\Big|_{s=0} . \end{aligned} \tag{31}$$

Let the operator Q be defined by

$$Qf(t) = \sum_\alpha a_\alpha \partial_s^\alpha f(t(s)t)\Big|_{s=0} ,$$

a right invariant differential operator on the group T in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii). □