Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: VITALI'S CONVERGENCE THEOREM ON TERM BY TERM INTEGRATION

Autor: Choksi, J. R.

Kapitel: 2. Vitali's convergence theorem

DOI: https://doi.org/10.5169/seals-65438

Fatou's proof of his lemma is very similar. It should be noted that Fatou's long paper is one of the most important of the century. For the first time the new theory of integration is applied to complex function theory; there are also fundamental applications to trigonometric series.

It is not until 1908, that DCT first appears in Lebesgue (1908) p. 9–10 [16] with a sketch of the proof; the same thing happens in Lebesgue (1909) [17] at the top of p. 50. In these papers Lebesgue seeks to apply his new results and finds BCT insufficient. In Lebesgue (1910) [18], in § 15 on page 375, the proof of DCT is given in more detail, still on a set of finite measure.

Sketch of his proof. Let $\varepsilon > 0$; since g is integrable on E, there exists a number $M > 0$, such that $\int_F g < \varepsilon$, where $F = \{g > M\}$; then $\int_F |f_n - f| < 2\varepsilon$, and on $E \backslash F$, the result follows by BCT.

Note that all the theorems so far have been stated and proved for sets E of *finite* measure. There does not seem at that time to have been much interest on anyone's part in extending the results and proofs for the case $m(E) = +\infty$. However, (excluding of course BCT) this is easily done.

2. Vitali's convergence theorem

In 1907, *before* Lebesgue announced DCT, there appeared a remarkable paper by G. Vitali [29], which, I feel, has not received its due, even from Hawkins. In it Vitali proves the following result:

Let E be a set of finite measure (finiteness is essential here). Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f finite a.e. Then f is integrable and $\int_F f_n \to \int_F f$, for every measurable subset F of E, if and only if the integrals $\int_A f_n$ are uniformly absolutely continuous (uniformly in n): given $\varepsilon > 0$, there exists $\delta > 0$, such that if $m(A) < \delta$, then $\left|\int_A f_n\right| < \varepsilon$ for all n.

This implies that $\int_A |f_n| < 2\varepsilon$. Vitali calls this *equi-absolutely continuous.* Note that this result generalizes at once to any finite measure space.

Vitali first proves that uniform absolute continuity is sufficient for $\int_F f_n \to \int_F f$ for all measurable subsets F of E.

Sketch of his proof. For $h \in \mathbf{N}$, let $G_h = \{|f_n| > 2^h \text{ for some } n\}$. If $\Gamma_h = E \backslash G_h$, then the sequence $\{f_n\}$ is uniformly bounded on Γ_h for every $h \in \mathbf{N}$. Thus on all measurable subsets of Γ_h, convergence of integrals follows by BCT. On the other hand, G_h is a decreasing sequence of sets and $m(\bigcap_h G_h) = 0$, so $m(G_h) \downarrow 0$. So for all sufficiently large h, the uniform absolute continuity condition implies that the contribution of the integrals over G_h is small.

Vitali next proves necessity of the uniform absolute continuity condition when the functions f_n are all non-negative.

Sketch of his proof. If $\int f_n$ are not uniformly absolutely continuous, then for some $\varepsilon > 0$, there exists for each $\delta > 0$, a measurable set F with $m(F) < \delta$ and $n \in \mathbf{N}$ with $\int_F f_n > \varepsilon$. Let $\delta_i > 0$ with $\Sigma \delta_i < \infty$. For each δ_i, there exists a measurable set $G_i \subseteq E$ and $n_i \in \mathbf{N}$ such that $m(G_i) < \delta_i$ and $\int_{G_i} f_{n_i} > \varepsilon$. Let $\Gamma_r = \bigcup_{i=r}^{\infty} G_i$. Then Γ_r decrease with r and $m(\Gamma_r) < \sum_{i=r}^{\infty} \delta_i \to 0$ as $r \to \infty$. For all $i \geq r$, $\int_{\Gamma_r} f_{n_i} > \varepsilon$. Since $f_{n_i} \to f$ a.e. as $i \to \infty$, $\int_{\Gamma_r} f_{n_i} \to \int_{\Gamma_r} f$ by hypothesis, and so for each r, $\int_{\Gamma_r} f \geq \varepsilon > 0$. Put $\Gamma = \bigcap_{r=1}^{\infty} \Gamma_r$. Then $m(\Gamma) = 0$, but $\int_{\Gamma} f \geq \varepsilon > 0$. Contradiction.

Finally Vitali proves necessity in the general case. If $f_n \to f$ a.e. on E and f_n are *completely integrable* on E, i.e. $\int_F f_n \to \int_F f$ for every measurable subset F of E, then $\int f_n$ are uniformly absolutely continuous, i.e. given $\varepsilon > 0$ there exists $\delta > 0$, such that if $m(A) < \delta$, then $\left| \int_A f_n \right| < \varepsilon$ for all $n \in \mathbf{N}$. (This is the deepest and hardest part of Vitali's paper, and is in a sense 'new' even after 93 years !)

Proof (Vitali). All the sets that occur in this proof will be measurable, even when this is not explicitly stated.

STEP I. If $f_n \geq 0$ for all $n \in \mathbf{N}$, we have already seen above that the result is true.

STEP II. Suppose now that $f > 0$ a.e. on E; we can assume that $f > 0$ on *all* of E. Note first that if $f_n \to f$ boundedly then BCT implies that $\int_E |f_n - f| \to 0$ as $n \to \infty$, and so given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $F \subseteq E$ and $n \geq N$, we have $\left| \int_F f_n - \int_F f \right| < \varepsilon$.

Let $G_n = \{0 < f_j < 2^n, \ \forall j \geq n\}$. $G_n \subseteq G_{n+1}$ for all n and $E = \bigcup_{n=1}^{\infty} G_n$. So $E \backslash G_n \downarrow \varnothing$ and $m(E \backslash G_n) \downarrow 0$.

Now given $\sigma > 0$, there exists $m(n, \sigma)$ such that if $\Gamma_n \subseteq G_n$ then $\left| \int_{\Gamma_n} f_j - \int_{\Gamma_n} f \right| < \sigma$ for all $j \geq m(n, \sigma)$. Let $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$. We can find a strictly increasing sequence of positive integers $n_1 < n_2 < \cdots < n_i < \cdots$ such that for every subset Γ_{n_i} of G_{n_i},

$$(\dagger) \qquad \left| \int_{\Gamma_{n_i}} f_j - \int_{\Gamma_{n_i}} f \right| < \varepsilon_i \quad \text{for } j \geq n_{i+1} .$$

For every positive integer $n > n_2$, there exists a unique $i \in \mathbf{N}$ such that $n_{i+1} \leq n < n_{i+2}$. For such n, put

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \in G_{n_i} \\ 0 & \text{if } x \in E \backslash G_{n_i} . \end{cases}$$

Let Γ be a subset of E and $\Gamma_{n_i} = G_{n_i} \cap \Gamma$. For fixed i,

$$\lim_{k \to \infty} \int_{\Gamma_{n_i}} g_k = \lim_{k \to \infty} \int_{\Gamma_{n_i}} f_k = \int_{\Gamma_{n_i}} f .$$

If $n_{i+1} \leq n < n_{i+2}$, then $\int_{\Gamma_{n_i}} g_n = \int_{\Gamma} g_n$, and so, using $(\dagger)$,

$$\lim_{n \to \infty} \int_{\Gamma} g_n = \lim_{i \to \infty} \int_{\Gamma_{n_i}} f = \int_{\Gamma} f = \int_{\Gamma} \lim g_n .$$

Thus g_n are completely integrable on E and since $g_n \geq 0$ for all n, it follows by Step I that $\int g_n$ are uniformly absolutely continuous on E. Put $\phi_n = f_n - g_n$. The ϕ_n are completely integrable on E. To complete the proof of Step II we must show that $\int \phi_n$ are uniformly absolutely continuous.

Observe that if $n_{i+1} \le n < n_{i+2}$, then $\phi_n(x) = 0$ for all $x \in G_{n_i}$. So for all $x \in E$, $\lim \phi_n(x) = 0$, and for any measurable subset $\Omega \subset E$, $\int_\Omega \phi_n \to 0$.

Suppose that $\int \phi_n$ are *not* uniformly absolutely continuous. Then there exists $\sigma > 0$, such that for all $\mu > 0$ and $N \in \mathbf{N}$, there exist $\Gamma \subseteq E$ with $m(\Gamma) < \mu$ and $n > N$ such that $\left|\int_\Gamma \phi_n\right| > \sigma$. Let $\eta_1, \eta_2, \ldots$ be > 0, and such that $\Sigma \eta_i < \frac{\sigma}{2}$. Let Γ_1 be a subset of E for which there exists $t_1 \in \mathbf{N}$ with $\left|\int_{\Gamma_1} \phi_{t_1}\right| > \sigma$. Since $\lim_{i\to\infty} \int_{G_{n_i}\cap\Gamma_1} \phi_{t_1} = \int_{\Gamma_1} \phi_{t_1}$, we can find $i_1 \in \mathbf{N}$ such that $\left|\int_{G_{n_{i_1}}\cap\Gamma_1} \phi_{t_1}\right| > \sigma$. Now there exists $\mu_1 > 0$ such that if $m(\Gamma) < \mu_1$, then $\left|\int_\Gamma \phi_{t_1}\right| < \eta_1$. By our assumption, there exists Γ_2 such that $m(\Gamma_2) < \mu_1$ and $t_2 \ge n_{i_1+1}$ such that $\left|\int_{\Gamma_2} \phi_{t_2}\right| > \sigma$. By the same reasoning, there exists i_2 (necessarily $> i_1$) such that $\left|\int_{G_{n_{i_2}}\cap\Gamma_2} \phi_{t_2}\right| > \sigma$.

Now there exists $\mu_2 > 0$ such that if $m(\Gamma) < \mu_2$, then

$$\left|\int_\Gamma \phi_{t_j}\right| < \eta_2,\ j = 1, 2.$$

There exist a subset Γ_3 with $m(\Gamma_3) < \mu_2$ and $t_3 \ge n_{i_2+1}$, such that $\left|\int_{\Gamma_3} \phi_{t_3}\right| > \sigma$. Again we can find $i_3 \in \mathbf{N}$ such that $\left|\int_{G_{n_{i_3}}\cap\Gamma_3} \phi_{t_3}\right| > \sigma$. Now there exists $\mu_3 > 0$ such that if $m(\Gamma) < \mu_3$, then

$$\left|\int_\Gamma \phi_{t_j}\right| < \eta_3\,,\ j = 1, 2, 3.$$

Continue in this way to obtain an increasing sequence $i_j, \mu_j > 0$, and $t_j \ge n_{i_{j-1}+1}$ and sets Γ_j such that $m(\Gamma_j) < \mu_{j-1}$ and $\left|\int_{\Gamma_j} \phi_{t_k}\right| < \eta_{j-1}$, $k = 1, 2, \ldots, j-1$, but $\left|\int_{G_{n_{i_j}}\cap\Gamma_j} \phi_{t_j}\right| > \sigma$. Let

$$\Omega_j = \{x \in G_{n_{i_j}} \cap \Gamma_j \text{ such that } \phi_{t_j}(x) \neq 0\}.$$

We claim the sets Ω_j are disjoint. Since $t_j \geq n_{i_{j-1}+1}$ we have $\phi_{t_j} = 0$ on $G_{n_{i_{j-1}}}$, and so $\phi_{t_{j+1}} = 0$ on $G_{n_{i_j}}$. So $\Omega_{j+1} \cap G_{n_{i_j}} = \varnothing$. Since G_n increase with n, i.e. $G_n \subseteq G_{n+1}$ for all n, it follows that $\Omega_{j+1} \cap \Omega_k = \varnothing$ if $k = 1, 2, \ldots, j$, and so Ω_j are all disjoint. Put $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Then since $\phi_{t_j} = 0$ on $\Omega_1, \ldots, \Omega_{j-1}$,

$$\int_{\Omega} \phi_{t_j} = \int_{\Omega_j} \phi_{t_j} + \sum_{h=1}^{\infty} \int_{\Omega_{j+h}} \phi_{t_j}.$$

But $\left| \int_{\Omega_j} \phi_{t_j} \right| > \sigma$ and $\left| \int_{\Omega_{j+h}} \phi_{t_j} \right| < \eta_{j+h-1}$, so $\left| \sum_{h=1}^{\infty} \int_{\Omega_{j+h}} \phi_{t_j} \right| < \frac{\sigma}{2}$. Hence $\left| \int_{\Omega} \phi_{t_j} \right| > \frac{\sigma}{2}$ for all j. But since ϕ_n are completely integrable and $\phi_n(x) \to 0$ for all $x \in E$ it follows that $\int_{\Omega} \phi_{t_j} \to 0$ as $j \to \infty$. This is a contradiction, completing the proof of Step II.

STEP III. In the general case, put $g_n = f_n - f + 1$. Then $g_n \to 1$ a.e. on E, and g_n is completely integrable. Hence as seen in Step II, the result is true for the g_n, i.e., $\int g_n$ are uniformly absolutely continuous. Hence the same holds for $\int f_n$, completing the proof.

COMMENTS

The concept of complete integrability of a sequence is weaker than weak sequential convergence in L^1. Vitali was of course, in 1907, unaware of L^1 convergence (strong or weak) and its significance. Using Vitali's proof that complete integrability implies uniform absolute continuity, we see that it also implies convergence in L^1. But a *direct* proof that complete integrability implies L^1 convergence (without using Vitali's result) seems hard.

A feature of this remarkable paper is that all the results are stated and proved in terms of series of functions rather than sequences; thus one has to realize that a "series all of whose partial sums are non-negative" corresponds to a sequence of non-negative terms, and is *not* to be confused with "a series of non-negative terms", which corresponds, of course, to an increasing sequence of non-negative terms ! Vitali's theorem is obviously a generalization of BCT. He remarks that Beppo Levi's MCT follows from it. Of course, so also does DCT, but Vitali did not know about DCT at the time.

The history and use of this result between 1907 and 1939 is something I would like to know more about! Hawkins [12] mentions the paper, but quotes only what I consider to be a much less important result at the end of the paper. In a footnote on p. 50 of his 1909 paper [17], Lebesgue says that DCT and MCT are special cases of Vitali's convergence theorem. He also states that DCT can be extended to sets of infinite measure. On p. 365 of the 1910 paper [18] he again refers to Vitali's Theorem, saying that it gives a necessary and sufficient condition for term by term integration. In *Leçons II* [19] (p. 131) Lebesgue merely refers to the paper: «M. Vitali a écrit sur ce sujet un très important Mémoire, que je ne puis ici que signaler»; this is just before he gives DCT. In 1913, Camp, in a rather messy paper [2], gives a generalization of Vitali's theorem to several variables.

In 1915, de la Vallée Poussin, wrote a long paper [27] entitled «Sur l'intégrale de Lebesgue»; this article is complementary to his book *«Intégrales de Lebesgue, fonctions d'ensembles, classes de Baire»* [28] written at about the same time. In the paper, in the section on convergence theorems, de la Vallée Poussin discusses Vitali's work, and in the proof of Theorem 4 on p. 448–450 he simplifies considerably the hard part of Vitali's proof; we give a sketch of his argument.

It is clearly sufficient to prove that if $f_n \to 0$ on E, and $\int f_n$ are *not* uniformly absolutely continuous on E, then there exists $F \subseteq E$ such that $\int_F f_n \not\to 0$. (We know this is true if $f_n \geq 0$ on E.) Let A_m be a sequence (to be chosen later) such that $0 < A_m < A_{m+1}$ for all $m \in \mathbf{N}$, and $A_m \to +\infty$, and let $E_m = \{x \in E : |f_n(x)| > A_m \text{ for some } n \in \mathbf{N}\}$. Note that the measure of E_m tends to zero.

Let $\varepsilon = \limsup \int_E |f_n|$; $\varepsilon > 0$, else $\int |f_n|$ are uniformly absolutely continuous. Let $\omega > 0$ with $\omega < \varepsilon/6$. It is fairly easy to choose A_m so that for each $m \in \mathbf{N}$, there exists $n \in \mathbf{N}$ such that the following three inequalities are satisfied:

$$(\ddagger) \qquad \int_{E \setminus E_m} |f_n| < \omega\,, \quad \int_{E_m} |f_n| > \varepsilon - \omega\,, \quad \int_{E_{m+1}} |f_n| < \omega\,.$$

This is done inductively: for each m, we can find n so that the first two inequalities are satisfied and then choose $A_{m+1} (> A_m)$ depending on n so that the third is satisfied. Further we can choose n increasing and $\to \infty$ with m.

($\ddagger$) implies that $\int_{E_m \setminus E_{m+1}} |f_n| > \varepsilon - 2\omega$, and so there exists $F_m \subseteq E_m \setminus E_{m+1}$ so that $\left| \int_{F_m} f_n \right| > \frac{1}{2}(\varepsilon - 2\omega)$. Put $F = \bigcup_{m=1}^{\infty} F_m$ (disjoint union). Note that

$F_i \subseteq E \backslash E_m$ for $i = 1, \ldots, m-1$ and $F_h \subseteq E_{m+1}$ for $h = m+1, m+2, \ldots$. Hence for each m, and the corresponding n,

$$\begin{aligned}\left|\int_F f_n\right| &\geq \left|\int_{F_m} f_n\right| - \sum_{i=1}^{m-1}\left|\int_{F_i} f_n\right| - \sum_{h=m+1}^{\infty}\left|\int_{F_h} f_n\right| \\ &\geq \left|\int_{F_m} f_n\right| - \int_{E\backslash E_m}|f_n| - \int_{E_{m+1}}|f_n| \\ &> \frac{\varepsilon - 2\omega}{2} - \omega - \omega = \frac{\varepsilon}{2} - 3\omega,\end{aligned}$$

which is positive since $\omega < \varepsilon/6$. Hence $\int_F f_n \nrightarrow 0$, completing the proof.

The argument is very similar to Step 1 in Hahn's proof of the Vitali-Hahn-Saks Theorem [10] given in §3, and it is at least conceivable that Hahn got the initial impetus for his proof from de la Vallée Poussin's paper. In [27], Theorem 5 on p. 450, de la Vallée Poussin shows that uniform absolute continuity of $\int f_n$ on a space of finite non-atomic measure is equivalent to: *Given $\varepsilon > 0$, there exists $K > 0$, such that for all $n \in \mathbf{N}$,* $\int_{\{|f_n|>K\}} |f_n| < \varepsilon$.

This was rediscovered by Doob [3] 24 years later, the new criterion was called *uniform integrability*, and used extensively by Doob in his study of martingales. In 1918, H. Hahn [9, p. 1774] showed, using Vitali's result, that complete integrability implies strong L^1 convergence: this shows that Hahn was aware of Vitali's paper. However de la Vallée Poussin's paper seems to be virtually unknown! I learnt about it from the excellent set of bibliographical references on p. 223 of Hahn and Rosenthal [11]. Nagumo [21] discusses the theorem with reference to Vitali, uses it, and gives a necessary and sufficient condition for uniform absolute continuity. Vitali himself does not seem to have worked further on this subject. See the biographical article by A. Tonolo [26].

Among well-known books on real analysis written before World War II only Hobson [13] refers to Vitali's paper on p. 296–299. Hobson also has what is probably the first attempt to generalize Vitali's result to sets of infinite measure; these are all, in my opinion, somewhat artificial. Since the fifties some books on analysis and/or probability have included the concept of uniform absolute continuity or uniform integrability, but often without any mention of Vitali. Also, where there is a reference to Vitali, the result attributed to him is often the equivalence of uniform absolute continuity and strong L^1

convergence, which follows from the easy part of Vitali's work, whereas complete integrability is not mentioned. Rudin, *Real and Complex Analysis* [24], is an exception – in all three editions; however, in the first edition the Vitali convergence theorem is given, by the third edition this has changed to the Vitali-Hahn-Saks theorem. Dunford-Schwartz [6] has a comprehensive account in Chapters III and IV. Unfortunately there is a slip in the statement of Vitali's convergence theorem on p. 234.

3. THE VITALI-HAHN-SAKS THEOREM

Vitali's convergence theorem is regarded as the origin of this theorem. It was first stated and proved by H. Hahn [10] in 1922. Hahn's statement and proof follow. (Both this result and Corollary 2 are referred to as "The Vitali-Hahn-Saks Theorem". The result is obviously stronger than Vitali's convergence theorem.)

THEOREM (H. Hahn [10] Thm. XXI, pages 45–50). *If* $m(E) < \infty$, f_n *integrable on* E, *and for each measurable* $F \subseteq E$, $\lim_{n\to\infty} \int_F f_n$ *exists and is finite, then* $\int f_n$ *are uniformly absolutely continuous.*

Proof. Again, all the sets that occur in this proof will be measurable. Suppose the integrals are not uniformly absolutely continuous. Then there exists $\varepsilon > 0$ with the property that for each $N \in \mathbf{N}$ and $\sigma > 0$ there is a measurable set Z with $m(Z) < \sigma$ and $n_0 > N$ with $\int_Z |f_{n_0}| > \varepsilon$. By considering the sets where $f_{n_0} \geq 0$ and $f_{n_0} \leq 0$, we obtain for each $N \in \mathbf{N}$, a set M with $m(M) < \sigma$ and $n_0 > N$ with $\left|\int_M f_{n_0}\right| > \frac{\varepsilon}{2}$.

STEP 1. We show that there exists a sequence of pairwise disjoint sets M_ν and an increasing sequence of positive integers n_ν such that

$$\left|\int_{M_\nu} f_{n_\nu}\right| \geq \frac{\varepsilon}{2} \text{ for all } \nu \in \mathbf{N}.$$

We start by choosing a proper subset Z_1 of E and $n_1 \in \mathbf{N}$ such that $\left|\int_{Z_1} f_{n_1}\right| > \frac{\varepsilon}{2}$. We observe that there exists $\sigma > 0$ sufficiently small so that