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5. Inversion and Modularity

Since $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \subseteq \mathrm{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{V}^*)$, the inverse ϕ^{-1} of a nondegenerate $\phi \in \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ is well defined and lies in $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$. By Cramer's rule inversion is a rational map from $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ to $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$, more precisely there is a homogeneous polynomial map $P\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ such that $\phi^P\phi = \det(\phi) \cdot id_{\mathcal{V}}$. Viewing this as an identity of matrices with polynomial entries, one might cancel out the greatest common divisor of all occurring entries and get new polynomial maps $p\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ and $d\colon \mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}) \to \mathbf{Q}$ with $\phi^p\phi = d(\phi) \cdot id_{\mathcal{V}}$. The properties of the map p have not been studied in this generality. The aim here is to investigate the simplest case, where p is homogeneous of degree 1, i. e. a $\mathbf{Q}$-linear map ι, as it is called in the sequel. Of course, the same analysis can be done with $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$. The question whether such a ι is an equivalence, will be treated later in this section.

Definition 5.1. Let R be one of $\mathbf{Z}$ or $\mathbf{Q}$. Then $\mathrm{Bil}_{\Lambda_R}(L_R)$ is called *special* if there is an R-linear map $\iota\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to \mathrm{Bil}_{\Lambda_R}(L_R^*)$ and a quadratic form $q\colon \mathrm{Bil}_{\Lambda_R}(L_R) \to R$ such that for any nondegenerate $\phi \in \mathrm{Bil}_{\Lambda_R}(L_R)$ one has $\phi^\iota\phi = q(\phi)\, id_{L_R}$. Analogous definitions hold for $\mathrm{Bil}^+_{\Lambda_R}(L_R)$

Example 5.2.

(i) One-dimensional lattices of covariant forms are special for trivial reasons.

(ii) If $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ is two-dimensional, then it is special. This is because $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})$-module and for two-dimensional algebras $\mathcal{B}$ one has a canonical automorphism κ of $\mathcal{B}$ such that $b^\kappa = n(b)b^{-1}$ for all $b \in \mathcal{B}^*$, where $n\colon \mathcal{B} \to F$ is the norm map with respect to the regular representation. (Note that $Z(\mathcal{A}) = \mathrm{End}_{\mathcal{A}}(\mathcal{V})$ in the present situation.)

(iii) If $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ is two-dimensional then it is special. This is because $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ can be viewed as a free $Z(\mathcal{A})^+$-module, where

$$Z(\mathcal{A})^+ := \{\varphi \in Z(\mathcal{A}) \mid \varphi^\circ = \varphi\}.$$

Here are some more interesting examples.

Proposition 5.3. *Let* $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong K^{2\times 2}$ *with* $K \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$. *Then* $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ *is special. In the first two cases also* $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ *is special.*

Proof. Define $\mathcal{E} := \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong (e\mathcal{A}e)^{k\times k}$, where $e = e^\circ$ is a primitive $^\circ$-invariant idempotent of $\mathcal{A}$ and k is defined by $\mathcal{V} \cong (e\mathcal{A})^k$. In particular, the positive involution $^\circ$ on $\mathcal{A}$ induces a positive involution $^\bullet$ on $\mathcal{E}$, $(a_{ij})^\bullet := (a_{ij}^\circ)^{tr}$, such that $\mathrm{Bil}_{\mathcal{A}}^+(\mathcal{V})$ can be identified with the subspace $\mathcal{E}^+$ of the symmetric elements in the algebra $(\mathcal{E}, ^\bullet)$ with involution. It suffices to prove that there exists a $\mathbf{Q}$-vector space automorphism of $\mathcal{E}^+$, also denoted by ι, and a $\mathbf{Q}$-valued quadratic form on $\mathcal{E}^+$, also denoted by q, such that $\phi^\iota\phi = q(\phi)\,1_{\mathcal{E}}$.

(i) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{R}^{2\times 2}$. Then $\mathcal{E}$ is a quaternion algebra over $\mathbf{Q}$. Denote its canonical involution by ω' and its reduced norm by n. Clearly, n is a quadratic form and $\omega'(\phi)\,\phi = n(\phi)\,1$ holds for all elements $\phi \in \mathcal{E}$. With $\iota := \omega'_{|\mathcal{E}^+}$ and $q := n_{|\mathcal{E}^+}$ one gets the desired formula.

(ii) Let $\mathbf{R}\otimes_{\mathbf{Q}}\mathcal{E} \cong \mathbf{C}^{2\times 2}$. Then $\mathcal{E}$ is a quaternion algebra over the imaginary quadratic number field $Z := Z(\mathcal{A})$. Denote its canonical involution by ω' and its reduced norm by n. The involution $^\bullet$ induces the nontrivial Galois automorphism of $(Z/\mathbf{Q})$, and therefore one checks quite easily, using [Scha85] Theorem 11.2 (ii) of Chapter 8, that the norm n maps $\mathcal{E}^+$ into $\mathbf{Q}$. Now one argues as in (i).

(iii) Let $\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{E} \cong \mathbf{H}^{2\times 2}$. Then $\mathcal{E} \cong D^{2\times 2}$, where D is a positive definite quaternion algebra over $\mathbf{Q}$ (with canonical involution ω'). Indeed, $\mathcal{E}$ carries an involution of the first kind and hence cannot be of index 4. Since $^\bullet$ is a positive involution one sees from the proof of Theorem 13.3 of Chapter 8 in [Scha85] that $x^\bullet = f^{-1}\overline{x}^{tr} f$ for all $x \in \mathcal{E}$, where $f = \overline{f}^{tr} \in \mathcal{E}^*$ and $\overline{(x_{ij})} = (\overline{x_{ij}})$ for all $(x_{ij}) \in D^{2\times 2} \equiv \mathcal{E}$. If $(x_{ij}) \in \mathcal{E}$ is symmetric with respect to $^{-tr}$ one checks

$$(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} \quad \text{with } \overline{x_{ii}} = x_{ii} \text{ for } i = 1,2$$

$$\text{and} \quad \begin{pmatrix} x_{22} & -x_{12} \\ -\overline{x_{12}} & x_{11} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \overline{x_{12}} & x_{22} \end{pmatrix} = (x_{22}\,x_{11} - x_{12}\,\overline{x_{12}})\,1_{\mathcal{E}}\,.$$

This is the desired formula for $f = 1_{\mathcal{E}}$. In the general case, note that $x \in \mathcal{E}^+$ if and only if fx is symmetric with respect to $^{-tr}$ and apply the above formula to fx.

(iv) The remaining two cases for $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ are treated similarly, like (i) and (ii) with $\mathcal{E}^+$ replaced by $\mathcal{E}$. □

The question immediately arises, whether the map ι of Definition 5.1 is or can be extended to an equivalence of $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})$ onto $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V}^*)$. This is clearly the case for two-dimensional $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$. It may fail for two-dimensional $\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ with four-dimensional commutative $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ for the simple reason that the nontrivial automorphism of the real quadratic subfield does not necessarily extend to the whole of $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$. For $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{R}^{2\times 2}$ one gets a nice canonical answer, cf. Proposition 5.4 below. For $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{C}^{2\times 2}$ the answer is still positive, but the proof is computational and we omit it. Finally, for $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{H}^{2\times 2}$ the map ι no longer extends to an equivalence.

PROPOSITION 5.4. *Let* $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong\mathbf{R}^{2\times 2}$. *Then any nonzero* $\psi\in\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ *defines an equivalence* $\mathrm{Bil}_{\mathcal{A}}(\mathcal{V})\to\mathrm{Bil}_{\mathcal{A}}(\mathcal{V}^*):\phi\mapsto\psi\phi\psi^{tr}$ *which restricts to a map* $\iota\colon\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})\to\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V}^*)$ *with the properties described in Proposition 5.3.*

Proof. If $\mathcal{V}$ is a simple $\mathcal{A}$-module, obviously any nonzero element of $\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ is invertible if viewed as an $\mathcal{A}$-homomorphism from $\mathcal{V}^*$ to $\mathcal{V}$. Otherwise, $\mathcal{V}\cong\mathcal{V}_0\oplus\mathcal{V}_0$ for some simple $\mathcal{A}$-module $\mathcal{V}_0$. Any $\mathcal{A}$-isomorphism $\mathcal{V}_0\to\mathcal{V}_0^*$ gives rise to an invertible element of $\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V})$, which therefore consists of 0 and invertible elements, since it is one-dimensional. One easily checks that any nonzero $\psi\in\mathrm{Bil}^-_{\mathcal{A}}(\mathcal{V}^*)$ leads to an equivalence, whose associated isomorphism $\mathrm{End}_{\mathcal{A}}(\mathcal{V}\oplus\mathcal{V}^*)\to\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*\oplus\mathcal{V})$ is induced by conjugation with $\mathrm{diag}(-\psi^{-1},\psi)$. Finally, for any $\phi\in\mathrm{Bil}^+_{\mathcal{A}}(\mathcal{V})$ one has $\phi(\psi\phi\psi^{tr})=q(\phi)id_{\mathcal{V}}$ with $q(\phi):=n(\psi\phi)$, where n is the reduced norm map of the quaternion algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$. This is so, since $\phi(\psi\phi\psi^{tr})=-(\phi\psi)^2$ and $\phi\psi$ lies in $\mathrm{End}_{\mathcal{A}}(\mathcal{V}^*)$ and is of trace zero by $tr(\phi\psi)=tr((\phi\psi)^{tr})=tr(-\psi\phi)=-tr(\phi\psi)$. □

The next result normalizes ι and interprets it in the integral environment of $\mathrm{Bil}^+_{\Lambda}(L)$.

THEOREM 5.5. *Let* $\mathbf{R}\otimes_{\mathbf{Q}}\mathrm{End}_{\mathcal{A}}(\mathcal{V})\cong K^{2\times 2}$ *with* $K\in\{\mathbf{R},\mathbf{C},\mathbf{H}\}$.

(i) *There is a unique* $\mathrm{Aut}(\mathrm{Bil}_{\Lambda}(L))$*-invariant quadratic form* $q\colon\mathrm{Bil}^+_{\Lambda}(L)\to\mathbf{Z}$ *such that the* $\gcd(q(\phi))$ *for* $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$ *is* 1, *and* $q(\phi)>0$ *for* $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$ *positive definite.*

(ii) *There is a unique constant* $c\in\mathbf{Z}$ *satisfying* $\det(\phi)=cq(\phi)^m$ *with* $m=2^{-1}\dim_{\mathbf{Q}}\mathcal{V}$ *for all* $\phi\in\mathrm{Bil}^+_{\Lambda}(L)$. *(Clearly* $c\geq 1$.*)*

(iii) *There is a unique* $\mathrm{Aut}(\mathrm{Bil}_{\Lambda}(L))$*-monomorphism* $\iota\colon\mathrm{Bil}^+_{\Lambda}(L)\to\mathrm{Bil}^+_{\Lambda}(L^*)$ *mapping positive definite forms on positive definite ones such that the image of* ι *is not contained in* $p\,\mathrm{Bil}^+_{\Lambda}(L^*)$ *for any integer* $p\geq 2$.

(iv) *There is a unique constant* $c_0 \in \mathbf{Z}$ *with* $\phi^\iota\phi = c_0 q(\phi) id_L$ *for all* $\phi \in \mathrm{Bil}_\Lambda^+(L)$*. Moreover* c *divides* c_0^n*, where* $n = \dim_{\mathbf{Q}} \mathcal{V}$*. (In fact* $\det(\phi^\iota) = c_0^n c^{-1} q(\phi)^m$ *for all* $\phi \in \mathrm{Bil}_\Lambda^+(L)$*.)*

(v) $\mathrm{Aut}\big(\mathrm{Bil}_\Lambda^+(L)\big) \leq \mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ *is a subgroup of finite index.*

Proof. Let $\mathrm{Bil}_\Lambda^+(L) = \langle \phi_1, \phi_2, \dots, \phi_d \rangle_{\mathbf{Z}}$ (with $d = 3$, 4, resp. 6 for $K = \mathbf{R}$, $\mathbf{C}$, resp. $\mathbf{H}$). Choose the isomorphism ι of Proposition 5.3 by multiplying with a suitable positive rational number such that $\mathrm{Bil}_\Lambda^+(L)$ is mapped into $\mathrm{Bil}_\Lambda^+(L^*)$ but not into a proper multiple of $\mathrm{Bil}_\Lambda^+(L^*)$. After rescaling q of Proposition 5.3 appropriately, one gets a quadratic form $\widetilde{q} \in \mathbf{Z}[x_1, \dots, x_d]$ with

$$\Big(\sum_{i=1}^d x_i\phi_i^\iota\Big)\Big(\sum_{i=1}^d x_i\phi_i\Big) = \widetilde{q}(x_1, \dots, x_d)\, id_L\,.$$

Since $\mathbf{Z}[x_1, \dots, x_d]$ is a unique factorization domain, one obtains a constant c_0 and a quadratic form q as required in (i) and (iv). Also by taking determinants, the unique factorization property yields $\det(\phi) = cq(\phi)$ with a unique integer c dividing c_0^n. Since $\det(g\phi) = \det(g)^2 \det(\phi) = \det(\phi)$ for $g \in N(L)$, one sees that q is $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$-invariant, at least up to sign. And since the action respects positive definiteness, one gets invariance. One clearly has $(g\phi)^\iota = g^{-tr}\phi^\iota$ for all $g \in N(L)$ and all $\phi \in \mathrm{Bil}_\Lambda^+(L)$ of nonzero determinant. But since all other elements of $\mathrm{Bil}_\Lambda^+(L)$ are rational linear combinations of these, one obtains the equation for all $\phi \in \mathrm{Bil}_\Lambda^+(L)$.

To prove (v) we first note that, by a standard Lie group argument, the group S of norm 1 units of $\mathrm{End}_{\mathcal{A}}(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V})$ is mapped onto the 1-component of $\mathrm{O}(\mathrm{Bil}_{\mathbf{R}\otimes\Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$. Also it is well known that the subgroup Γ of norm 1 elements of $\mathrm{End}_\Lambda(L)^*$ (which is clearly of finite index in $N(L)$) has finite covolume in S. This implies that $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ is of finite covolume in $\mathrm{O}(\mathrm{Bil}_{\mathbf{R}\otimes\Lambda}^+(\mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}), q)$ and therefore of finite index in $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$.

It follows from (v) and the fact that the signature of q is $(1, d-1)$ that $\mathrm{Aut}(\mathrm{Bil}_\Lambda^+(L))$ acts absolutely irreducibly on $\mathrm{Bil}_\Lambda^+(L)$. This again implies that the invariant quadratic form q is unique up to rational multiples, i. e. unique with the properties specified in (i). It also implies the uniqueness of ι in (iii). The uniqueness of the constants c_0 and c now follows from the considerations at the beginning of the proof. □

The corresponding results for the other examples given in Example 5.2 are left as exercises to the reader, who should note however that the action of $\mathrm{O}(\mathrm{Bil}_\Lambda^+(L), q)$ on $\mathrm{Bil}_\Lambda^+(L)$ need not be absolutely irreducible any more.

The next topic it to set the concepts of this chapter into relation with modular lattices as introduced by Quebbemann in [Que95]; cf. also [SSch98] and [Ple98] for surveys.

DEFINITION 5.6.

(i) $\phi \in \mathrm{Bil}_\Lambda^+(L)$ is said to be *k-modular*, for $k \in \mathbf{Z}$, if $(L^\sharp, k\phi)$ is isometric to (L, ϕ), where $L^\sharp = \{l \in \mathcal{V} \mid \phi(l, L) \subseteq \mathbf{Z}\}$. (Note the Gram matrix of ϕ on $L^\sharp$ is inverse to the Gram matrix on L if one chooses the bases dual to each other.)

(ii) $\mathrm{Bil}_\Lambda^+(L)$ is called *modular* if $\mathrm{Bil}_\Lambda^+(L)$ is special by the maps $\iota\colon \mathrm{Bil}_\Lambda^+(L) \to \mathrm{Bil}_\Lambda(L^*)$ and $q\colon \mathrm{Bil}_\Lambda^+(L) \to \mathbf{Z}$, cf. Definition 5.1, such that ι is (the restriction to $\mathrm{Bil}_\Lambda^+(L)$ of) an induced equivalence; cf. Definition 4.3.

Clearly, if $\mathrm{Bil}_\Lambda^+(L)$ is modular, each nondegenerate $\phi \in \mathrm{Bil}_\Lambda^+(L)$ is $c_0 q(\phi)$-modular with c_0 as in Theorem 5.5, and the isometries are all given by the same map. Some examples of two-dimensional modular lattices of covariant forms have already been investigated in the literature, cf. e. g. [Neb98b] where even the Hermite function was discussed for some examples or [Neb96a], where the extremal 3-modular lattice in dimension 24 was discovered. Here the main issue concerns the cases with $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2\times 2}$ or $\mathbf{C}^{2\times 2}$, since $\mathbf{H}^{2\times 2}$ cannot occur. Example 6.6 (i) provides an example where $\mathrm{Bil}_\Lambda^+(L)$ is special without being modular. It should be emphasized that induced equivalence between $\mathrm{Bil}_\Lambda(L)$ and $\mathrm{Bil}_\Lambda(L^*)$ is not an uncommon phenomenon. For instance it occurs whenever L and L^* are Λ-isomorphic. That the induced equivalence is ι, is rather rare.

PROPOSITION 5.7. *Let* $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{R}^{2\times 2}$ *and assume* $\mathrm{Bil}_\Lambda^-(L) = \mathbf{Z}\psi_1$ *and* $\mathrm{Bil}_\Lambda^-(L^*) = \mathbf{Z}\psi_2$ *with* $\psi_1\psi_2 = e \cdot id_L$ *for some natural number* e.

(i) *If* $e = 1$ *then* $\mathrm{Bil}_\Lambda(L)$ *is modular, with* ι *induced by* ψ_2.

(ii) *If* ψ_1 *and* ψ_2 *do not have the same elementary divisors, then* $\mathrm{Bil}_\Lambda^+(L)$ *is not modular.*

(iii) *If* $e^{\dim(L)} \neq \det(\psi_2)^2$ *then* $\mathrm{Bil}_\Lambda^+(L)$ *is not modular.*

Proof. (i) This follows along the lines of Proposition 5.4. That $\mathrm{Bil}_\Lambda(L)$ is mapped onto $\mathrm{Bil}_\Lambda(L^*)$ follows from the fact that $\det(\psi_2) = \pm 1$.

(ii) This is because induced equivalence respects elementary divisors.

(iii) This can be derived from (ii) by taking determinants. It can also be obtained from the observation that ψ_2 induces $e \cdot \iota$. □

EXAMPLE 5.8.

(i) Of the four irreducible Bravais groups of degree 8 whose commuting algebra is a nonsplit rational quaternion algebra (ramified at 2 and 3), cf. [Sou94], the e of Proposition 5.7 is 1, 2, 3 and 6. In all cases $\mathrm{Bil}_\Lambda^+(L)$ is modular and c_0 is equal to 1. In [Neb99] the Hermite function on the fundamental domains for these cases is plotted.

(ii) In Example 2.2 (ii), choose f_0 to be m-modular for some natural number m. Then $\mathrm{Bil}_\Lambda^+(L \oplus L)$ (in the notation of Example 2.2 (ii)) is modular, where the e of Proposition 5.7 is equal to m, as is c_0.

To test whether $\mathrm{Bil}_\Lambda^+(L)$ is modular, one can simply compute the images of a $\mathbf{Z}$-basis of $\mathrm{Bil}_\Lambda^+(L)$ under ι as described in Theorem 5.5 and find a simultaneous isometry of L to L^* (with respect to all of the forms, resp. their images). For this there is a powerful algorithm with implementation available, cf. [PlS97]. Instead of a whole basis, it is sometimes enough to look at one sufficiently general form; details on this will be given in a subsequent paper, as well as some examples with $\mathbf{R} \otimes_{\mathbf{Q}} \mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{C}^{2\times 2}$. One such example, involving the Leech lattice with $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ a non-split quaternion algebra over $\mathbf{Q}[\sqrt{-7}]$, is sketched in the last chapter of [Ple96].

6. SOME THREE-DIMENSIONAL LATTICES OF COVARIANT FORMS

This chapter is devoted to some examples in the case where $\mathrm{End}_{\mathcal{A}}(\mathcal{V}) \cong \mathbf{Q}^{2\times 2}$ and where the depth of $\mathrm{Bil}_\Lambda(L)$ is 0. The typical questions we try to answer are: how to relate the various invariants? are outer automorphisms possible? are modular lattices possible? how does the automorphism group of $\mathrm{Bil}_\Lambda^+(L)$ compare to the orthogonal group of $(\mathrm{Bil}_\Lambda^+(L), q)$? The simplest case is $\mathrm{End}_\Lambda(L) \cong \mathbf{Z}^{2\times 2}$, where all these questions can be answered.

THEOREM 6.1. *Let* $\mathrm{End}_\Lambda(L) \cong \mathbf{Z}^{2\times 2}$. *Then* $L = L_0 \oplus L_0$ *for some irreducible* Λ*-lattice* L_0. *Let* ϕ_0 *be the positive definite generator of* $\mathrm{Bil}_\Lambda^+(L_0)$. *Then* c, c_0, *and* q, *introduced in Theorem 5.5, are as follows.*

(i) *With respect to a suitable basis of* $\mathrm{Bil}_\Lambda^+(L)$, *the quadratic form* q *of Theorem 5.5 becomes* $xy - z^2$.

(ii) $c = \det(\phi_0)^2$.

(iii) c_0 *is the exponent of* $L_0^\sharp / L_0$, *i. e. the biggest elementary divisor of a Gram matrix of* ϕ_0.