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THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES

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ABSTRACT. Motivated by Elliott's K-theoretic classification of C^*-algebras of type AF, we compute the positive cone of the K-theory of some spaces. These include the spheres, the products of an odd-dimensional sphere by a sphere, the products of the 2-sphere by a sphere, and of the products $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This amounts to computing the geometric dimension of stable classes of complex vector bundles over these spaces. We establish a few general properties of the positive cone and of approximations to it, the γ-cone and the c-cone. We also get information on the Whitehead product structure in the homotopy groups of $BU(n)$. Moreover, we prove a "doubling formula" for Stirling numbers of the second kind.

1. INTRODUCTION

Let $\mathcal{G}(S)$ be the Grothendieck group completion of an abelian semigroup S, and let $\theta \colon S \longrightarrow \mathcal{G}(S)$ be the corresponding universal homomorphism. The image of θ, denoted by $\mathcal{G}_+(S)$, is a sub-semigroup of $\mathcal{G}(S)$. If S has a zero, in other words if it is an abelian monoid, then $\mathcal{G}_+(S)$ induces a translation invariant preordering on $\mathcal{G}(S)$ (i.e. a reflexive and transitive relation, but not necessarily antisymmetric). The elements of $\mathcal{G}_+(S)$ are called *positive* and $\mathcal{G}_+(S)$ is called the *positive cone* (see [Ell] and [Bla1]). The pair $(\mathcal{G}(S), \mathcal{G}_+(S))$ is an isomorphism invariant of S, and a basic question is: to what extent does this invariant characterize the abelian semigroup S?

The above notions are of interest in connection with the classification problem of C^*-algebras. For a unital C^*-algebra A, let $S = \mathrm{Proj}(A)$ be the abelian monoid of equivalence classes of projectors in the matrix algebra $\mathbf{M}_\infty(A)$. The K-theory of A, denoted by $K_0(A)$ or $K(A)$, is by definition

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the group $\mathcal{G}(\mathrm{Proj}(A))$. The positive cone in K-theory is $\mathcal{G}_+(\mathrm{Proj}(A))$ and it is denoted by $K_+(A)$. In [Ell], Elliott has put forward a program to classify a large class of unital C^*-algebras by invariants of a K-theoretic nature, such as $K(A)$, $K_+(A)$, [1] (the K-theory class of the unit), etc. (see also [Bla1]). For a compact Hausdorff space X, the algebra $C(X)$ of continuous complex valued functions on X is a unital C^*-algebra and its K-theory coincides with the topological K-theory $K^0(X)$ of the space X (according to the Swan-Serre theorem). In view of Elliott's program and to shed light on various conjectures, it is of great interest to determine for such spaces the positive cone $K_+(X) = \mathcal{G}_+(\mathrm{Proj}(C(X)))$. For any connected X, the preordering determined on $K^0(X)$ by the positive cone is an ordering, as is easily checked (see also p. 84 in [Rord]).

The problem of computing the positive cone of some spaces and in particular of spheres has been communicated to us by Alain Valette, after a question asked by G. A. Elliott in Oberwolfach.

These notes are organized as follows. In Section 2, we recall the basic facts from topological K-theory needed in the sequel. Among other things, we review γ-operations. The computation of these operations for even-dimensional spheres puts Stirling numbers of the second kind on stage. In Section 3, we define what we call the γ-cone and the c-cone (the latter is defined in terms of Chern classes), and we explain in what sense they are approximations of the positive cone. We illustrate by examples that the three notions of cones are different in general, although the γ-cone and the c-cone coincide for torsion-free spaces.

In Section 4, we compute the positive cones of the spheres, by using some standard homotopy theory. Section 5 is devoted to the naturality properties of the three cones. The positive cone of the products $S^n \times S^{2m-1}$ is computed in Section 6. The γ-cone of the products $S^{2n} \times S^{2m}$ is easily calculated in Section 7 by means of Chern classes. In that section, we also compute the positive cone of $S^2 \times S^{2n}$.

The Whitehead product structure on the homotopy of the classifying space $BU(n)$ is closely related to the problem of determining the positive cone of the product of two even-dimensional spheres, as is explained in Section 8. This allows us to improve slightly a result of Bott on this structure, and gives some precise information on the positive cone of such a product of spheres.

In Section 9, we perform the computation of the positive cones of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and of $S^6 \times S^8$. This is achieved by using some well-known results on the homotopy groups of unitary groups. In Section 10, we show that for spaces "with only one high-dimensional cell" the γ-cone is "blind" in some sense to be made precise there.

Section 11 is devoted to explicitly computing the γ-operations for the products $S^{2n} \times S^{2m}$. As a consequence of these calculations, we establish a "doubling-formula" for Stirling numbers of the second kind. Moreover, we are led to conjecture that the same formula holds for Stirling numbers of the first kind. (This has now been proved by Al Lundell; see Theorem 11.2.)

2. Preliminaries

We start by reviewing some topological K-theory. Our basic references are the books by Atiyah [Atiyah] and by Husemoller [Huse].

Let X be a *connected finite* CW-complex. (We assume all spaces and maps to be pointed.) For each $n \geq 0$, let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex n-plane vector bundles over X, and $\mathrm{Vect}(X)$ their disjoint union. There are well-known bijections

$$\mathrm{Vect}_n(X) \approx [X, BU(n)] \qquad (n \geq 0)$$

where $BU(n)$ is the classifying space of the unitary group $U(n)$ and $[\,.\,,\,.\,]$ stands for the set of homotopy classes of maps. For an n-plane vector bundle ξ over X, i.e. $\xi \in \mathrm{Vect}_n(X)$, we write $\mathrm{rk}(\xi) = n$ (it is the *rank* of ξ). The direct sum (also called Whitney sum) and the tensor product of vector bundles endow $\mathrm{Vect}(X)$ with a semiring structure. The K-theory of X is the ring $K(X)$, also denoted by $K^0(X)$, obtained by applying the Grothendieck construction to $\mathrm{Vect}(X)$, i.e. $K(X) = \mathcal{G}(\mathrm{Vect}(X))$. An element of $K(X)$ is sometimes called a *virtual vector bundle*. There is a ring isomorphism

$$K(X) \cong [X, \mathbf{Z} \times BU],$$

where BU is the infinite Grassmannian, i.e. the direct limit of the classifying spaces $BU(n)$. We identify both rings from now on. There is a canonical splitting $K(X) = \mathbf{Z} \oplus [X, BU] = \mathbf{Z} \oplus \widetilde{K}(X)$, where $\widetilde{K}(X) = \widetilde{K}^0(X)$ is the subring of *stable classes* of vector bundles, and $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ is represented by the n-dimensional trivial vector bundle over X. Clearly, the Grothendieck construction gives rise to maps $\theta\colon \mathrm{Vect}(X) \longrightarrow K(X)$ and $\theta_n\colon \mathrm{Vect}_n(X) \longrightarrow n \times \widetilde{K}(X)$ (by restriction of θ).