Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 47 (2001)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES

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Kapitel: 11. A "DOUBLING FORMULA" FOR STIRLING NUMBERS OF THE SECOND KIND

DOI: https://doi.org/10.5169/seals-65432

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dimensions corresponding to the gap. At first sight, one could think that the γ-cone is more powerful in this range. Unfortunately, this is not the case: we show that the γ-cone (or equivalently the γ-dimension function) is also "blind" in some sense. Here is the precise statement.

PROPOSITION 10.1. *Let Y be a connected finite CW-complex of dimension $\leq 2n$, and let $X = C_f = Y \cup_f e^{2n+2m}$ be the mapping cone of a map $f\colon S^{2n+2m-1} \longrightarrow Y$, with $m \geq 1$. Then, for $x \in \widetilde{K}(X)$, one has*

$$\gamma^{n+m}(x) = 0 \implies \gamma^{n+l}(x) = 0 \quad \textit{for all} \quad l = 1, \dots, m\,.$$

In other words, if $\gamma\text{-dim}(x) < n + m$, then $\gamma\text{-dim}(x) \leq n$.

Proof. By assumption, one has $H^k(X; \mathbf{Z}) = 0$ for $2n < k < 2n + 2m$ and $H^{2n+2m}(X; \mathbf{Z}) \cong \mathbf{Z}$. Let $x \in \widetilde{K}(X)$ such that $\gamma^{n+m}(x) = 0$. By Proposition 2.2, keeping the same notation, we have

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \dots, \bar{c}_{n+m}(x))\,,$$

and $0 = ch(\gamma^{n+m}(x)) = \bar{c}_{n+m}(x)$. Due to the "gap" in the cohomology of X, we find that, for $k > n$, we have

$$ch(\gamma^k(x)) = 0\,.$$

By the particular cohomological properties of X, the Chern character is injective for elements of filtration $> n$ in $\widetilde{K}(X)$ (see [AtHi]). Being zero or of filtration $\geq k$ (as Proposition 2.2 shows), $\gamma^k(x)$ has to vanish for $k > n$. This concludes the proof. □

11. A "DOUBLING FORMULA" FOR STIRLING NUMBERS OF THE SECOND KIND

In the present section, we calculate the γ-operations for the product $S^{2n} \times S^{2m}$. From this computation and Proposition 10.1, we deduce again the γ-cone, as appearing in Theorem 7.1. This example illustrates that computing the c-cone is in general easier than computing the γ-cone. On the other hand, the latter calculation leads to an interesting "doubling formula" for Stirling numbers of the second kind. We will also conjecture the analogous formula for Stirling numbers of the first kind.

Keeping notations as in Section 7, we have

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot x_1 \oplus \mathbf{Z} \cdot x_2 \oplus \mathbf{Z} \cdot x_1 x_2 .$$

We still assume $n \leq m$. Using the known γ-operations for even-dimensional spheres, one can easily calculate γ^k for $S^{2n} \times S^{2m}$: For $x = ax_1 + bx_2 + lx_1x_2$, one has clearly $\gamma^k(x) = \gamma^k(ax_1 + bx_2) + \gamma^k(lx_1x_2)$ and this allows one to compute

$$\gamma^{m+q}(x) = (-1)^{m+q-1}(m+q-1)! \cdot \left(l\, S(m+n,\, m+q) - ab \sum_{k=q}^{n} \frac{S(n,\, k)S(m,\, m+q-k)}{k\binom{m+q-1}{k}} \right) \cdot x_1x_2$$

for $q \geq 1$; in particular

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - ab\,(n-1)!\,(m-1)!) \cdot x_1x_2 .$$

For γ^m, we have to distinguish the case $n = m$ from the case $n < m$. One gets

$$\gamma^m(x) = (-1)^{m-1}a\,(m-1)! \cdot x_1 \;+\; (-1)^{m-1}b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1} \cdot (m-1)! \left(l\, S(2m,\, m) - ab \sum_{k=1}^{m-1} \frac{S(m,\, k)S(m,\, m-k)}{k\binom{m-1}{k}} \right) \cdot x_1x_2$$

when $n = m$, whereas

$$\gamma^m(x) = (-1)^{m-1}b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1}(m-1)! \cdot \left(l\, S(n+m,\, m) - ab \sum_{k=1}^{m-1} \frac{S(n,\, k)S(m,\, m-k)}{k\binom{m-1}{k}} \right) \cdot x_1x_2$$

when $n < m$.

We want to compute the γ-dimension of $x = ax_1 + bx_2 + lx_1x_2$. If $l = 0$, the result is clear. We can now assume that $l \neq 0$. If l is different from $ab\,(n-1)!\,(m-1)!/(n+m-1)!$, we see that $\gamma\text{-dim}(x) = n+m$. On the other side, if l has precisely this value, then $\gamma^m(x) \neq 0$, because in this case $b \neq 0$, and by Proposition 10.1 we get $\gamma\text{-dim}(x) = m$ precisely. This gives another proof of Theorem 7.1.

Let us now pass to the "doubling formula".

THEOREM 11.1. *Let $q \leq n \leq m$ be positive integers; then*

$$S(m+n, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{S(n, k)S(m, m+q-k)}{k\binom{m+q-1}{k}}.$$

We called this a "doubling formula" because, particularizing to $n = m$, we get an expression allowing one to compute $S(2n, n+q)$ in terms of the numbers $S(n, k)$ with $q \leq k \leq n-1$.

Proof. This is an immediate consequence of Proposition 10.1 and the above computations. □

An alternative proof would be to invoke Theorem 7.1 rather than Proposition 10.1.

After trying to verify on a computer the analogous formula for Stirling numbers of the first kind, namely

$$s(n, k) = \sum_{j=0}^{n-k} \binom{-k}{n-k+j}\binom{2n-k}{n+j} S(n-k+j, j),$$

we were led to conjecture it:

THEOREM 11.2. *Let $q \leq n \leq m$ be positive integers; then*

$$s(m+n, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{s(n, k)s(m, m+q-k)}{k\binom{m+q-1}{k}}.$$

We call it a "theorem", since, after we had informed him about Theorem 11.1 and our conjecture, Al Lundell sent us a proof of the latter. The elegant proof is "elementary" in the following sense: it uses only some basic formulas for Stirling numbers (such as generating functions) and a contour argument in the computation of an integral, but no K-theory. Moreover, his proof encompasses the Stirling numbers of both the first *and* the second kind in a unified way.

AKNOWLEDGEMENTS. The first named author wishes to thank the University of Neuchâtel, Switzerland, for its partial support and its hospitality during the preparation of this work.