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Artikel: ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS

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Kapitel: 3. Malcev's example

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3. Real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over $\mathbf{Q}$.
5. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two Lie algebras with a unique rational form up to isomorphism.
6. Let $\mathfrak{g}$ be a split real simple Lie algebra of type G_2, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathfrak{n}_+$ is isomorphic to $\mathfrak{g}_0$.

3. Malcev's example

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a $\mathbf{Q}$-isomorphism between $\mathfrak{g}_t$ and $\mathfrak{g}_s$. It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer $\mathfrak{c}$ of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by $x_3, \dots, x_6$.

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = a_{44}x_4 + \dots \end{cases}$$

We do not explicit the expressions for y_5, y_6. Here $y_1, \dots, y_6$ are basis elements of $\mathfrak{g}_s$ satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots ,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases} \tag{3.1}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned} \tag{3.2}$$

Let $u = a_{12}a_{33} + a_{11}a_{34}$, $v = a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33}$. In view of (3.1) and (3.2) we have

$$(a_{22}y_5 - a_{12}y_6)\,u/\Delta^2 + (a_{11}y_6 - a_{21}y_5)\,v/\Delta^2 = y_6\,,$$

whence

$$\begin{cases} va_{11} - ua_{12} = \Delta^2\,, \\ va_{21} - ua_{22} = 0\,. \end{cases} \tag{3.3}$$

It follows that

$$\begin{cases} u = a_{21}\Delta\,, \\ v = a_{22}\Delta\,. \end{cases} \tag{3.4}$$

In addition,

$$\begin{aligned} [y_2, y_3] &= a_{21}a_{33}[x_1, x_3] + a_{22}a_{33}[x_2, x_3] + a_{21}a_{34}[x_1, x_4] + a_{22}a_{34}[x_2, x_4] \\ &= a_{21}a_{33}x_6 + a_{22}a_{33}(x_5 + tx_6) + a_{21}a_{34}x_5 + a_{22}a_{34}x_6 \\ &= (a_{22}a_{33} + a_{21}a_{34})x_5 + (a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33})x_6 = y_5 + sy_6\,. \end{aligned} \tag{3.5}$$

Let $p = a_{22}a_{33} + a_{21}a_{34}$, $q = a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33}$. In view of (3.1), (3.5)

$$(a_{22}y_5 - a_{12}y_6)\,p/\Delta^2 + (a_{11}y_6 - a_{21}y_5)\,q/\Delta^2 = y_5 + sy_6\,.$$

This implies that

$$\begin{cases} qa_{11} - pa_{12} = s\Delta^2\,, \\ qa_{21} - pa_{22} = -\Delta^2\,. \end{cases} \tag{3.6}$$

Consequently,

$$\begin{cases} p = (sa_{21} + a_{11})\,\Delta\,, \\ q = (sa_{22} + a_{12})\,\Delta\,. \end{cases} \tag{3.7}$$

Substituting u, v, p, q by the expressions given in (3.4), (3.7) we conclude that

$$\begin{cases} a_{11}a_{34} + a_{12}a_{33} = a_{21}\Delta\,, \\ a_{11}a_{33} + a_{12}(a_{34} + ta_{33}) = a_{22}\Delta\,, \\ a_{21}a_{34} + a_{22}a_{33} = (a_{11} + sa_{21})\,\Delta\,, \\ a_{21}a_{33} + a_{22}(a_{34} + ta_{33}) = (a_{12} + sa_{22})\,\Delta\,. \end{cases} \tag{3.8}$$

The first and the third equations of (3.8) yield

$$\begin{cases} a_{34} = a_{21}a_{22} - a_{12}(a_{11} + sa_{21})\,, \\ a_{33} = a_{11}(a_{11} + sa_{21}) - a_{21}a_{22}\,. \end{cases} \tag{3.9}$$

The two remaining ones yield

$$(3.10)\qquad \begin{cases} a_{33} = a_{22}a_{22} - a_{12}(a_{12} + sa_{22}), \\ a_{34} + ta_{33} = a_{11}(a_{12} + sa_{22}) - a_{21}a_{21}, \end{cases}$$

whence

$$(3.11)\qquad \begin{cases} a_{11}^2 + sa_{11}a_{21} - a_{21}^2 = a_{22}^2 - sa_{22}a_{12} - a_{12}^2 \neq 0, \\ 2(a_{11}a_{12} - a_{21}a_{22}) + s(a_{11}a_{22} - a_{21}a_{12}) = t(a_{11}^2 + sa_{11}a_{21} - a_{21}^2). \end{cases}$$

Let

$$\begin{cases} x_{11} = a_{11} + sa_{21}/2, \\ x_{12} = a_{12} + sa_{22}/2, \\ x_{21} = a_{21}, \\ x_{22} = a_{22}. \end{cases}$$

The system (3.11) can be rewritten in the form

$$(3.12)\qquad \begin{cases} x_{11}^2 - (1 + s^2/4)x_{21}^2 = -(x_{12}^2 - (1 + s^2/4)x_{22}^2) \neq 0, \\ 2(x_{11}x_{12} - (1 + s^2/4)x_{21}x_{22}) = t(x_{11}^2 - (1 + s^2/4)x_{21}^2). \end{cases}$$

Thus we may conclude that $\mathfrak{g}_t \cong \mathfrak{g}_s$ if and only if (3.12) has a rational solution such that $x_{11}x_{22} - x_{12}x_{21} \neq 0$. We state the following lemma in order to obtain less sophisticated conditions on s, t.

LEMMA 3.1. *Let $s, t \in \mathbf{Q}$. Then two conditions are equivalent:*
i) *there exists a matrix*

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q})$$

such that x, y, z, w satisfy the system

$$(3.13)\qquad \begin{cases} x^2 - (1 + s^2/4)z^2 = -y^2 + (1 + s^2/4)w^2 \neq 0, \\ 2(xy - (1 + s^2/4)zw) = t(x^2 - (1 + s^2/4)z^2). \end{cases}$$

ii) *there exists $q \in \mathbf{Q}$ such that*

$$(3.14)\qquad (t^2 + 4)(s^2 + 4) = q^2.$$

Proof. Let $p = 1 + s^2/4$, $r = 1 + t^2/4$. The system (3.13) yields

$$(3.15)\qquad \begin{cases} x^2 + y^2 = p(z^2 + w^2), \\ 2xy - tx^2 = p(2zw - tz^2). \end{cases}$$

After the change of variables

$$x = x_0\,,\qquad y = \frac{1}{2}(y_0 + tx_0)\,,\qquad z = \frac{z_0}{p}\,,\qquad w = \frac{1}{2}(w_0 + \frac{t}{p}z_0)$$

the system (3.15) can be rewritten as

$$(3.16)\qquad \begin{cases} rx_0^2 + \frac{1}{4}y_0^2 = \frac{r}{p}z_0^2 + \frac{p}{4}w_0^2\,, \\ x_0y_0 = z_0w_0\,. \end{cases}$$

Geometrically, the system (3.16) defines the intersection I of two quadrics in the projective space $\mathbf{P}^3$. Let $\sigma\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ be the Segre map. In homogeneous coordinates $(a : b\,; \alpha : \beta)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, σ is defined by $x_0 = a\alpha$, $y_0 = b\beta$, $z_0 = a\beta$, $w_0 = b\alpha$, and the image $\sigma(\mathbf{P}^1 \times \mathbf{P}^1)$ is the zero locus of the polynomial $x_0y_0 - z_0w_0$.

It is not hard to verify that in coordinates $(a : b\,; \alpha : \beta)$ the preimage $\sigma^{-1}(I)$ is given by the following equation (corresponding to the first one of (3.16)):

$$(4ra^2 - pb^2)(p\alpha^2 - \beta^2) = 0.$$

Thus $\sigma^{-1}(I)$ is the union of two pairs of lines (over $\mathbf{R}$). The second pair defined by the equation $p\alpha^2 - \beta^2 = 0$ yields $xw - zy = \det(M) = 0$. It follows that (3.15) has a rational solution if and only if the equation $4ra^2 - pb^2 = 0$ has one, i.e., $p/4r$ is the square of a rational number. This is equivalent to (3.14). Note that the condition $x^2 - pz^2 \neq 0$ in (3.13) is not very restrictive. This completes the proof of the lemma and of Theorem 1.

COROLLARY 3.2. *There are infinitely many non-isomorphic Lie algebras of the type* $\mathfrak{g}_s$ *over* $\mathbf{Q}$.

Proof. Let $s_1 = p_{11}$ be an odd prime. Consider $s_1^2 + 4 = p_1^2 + 4 = p_{21}^{n_{21}} \dots$ It is clear that $s_1^2 + 4$ is not a square (this means that at least one of the n_{2j} is odd) and is not divisible by p_{11}, whence all the $p_{2j} \neq p_{11}$. Let $s_2 = p_{11}p_{21} \dots$. It follows that

$$s_2^2 + 4 = p_{31}^{n_{31}} \dots$$

is not a square and is not divisible by p_{ij} where $i \leq 2$. Then we set $s_3 = p_{11}p_{21} \dots p_{31} \dots$ and so on.

In such a way we obtain an infinite sequence of numbers s_1, $s_2 \dots$ Let $i < j$. Note that $(s_i^2 + 4)(s_j^2 + 4) \neq q^2$, $q \in \mathbf{Q}$. Indeed, by the construction $(s_i^2 + 4)$ is divisible by some p and not divisible by p^2. Also, p divides s_j. Consequently, it does not divide $s_j^2 + 4$. This means that $(s_i^2 + 4)(s_j^2 + 4)$ is divisible by p but not by p^2, and this completes the proof.