Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 48 (2002)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

Autor: Kapovich, Ilya

Kapitel: 1. Introduction

DOI: https://doi.org/10.5169/seals-66081

L'Enseignement Mathématique, t. 48 (2002), p. 359–375

THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

by Ilya KAPOVICH

ABSTRACT. We show that if H is a quasiconvex subgroup of infinite index in a nonelementary hyperbolic group G then the Schreier coset graph for G relative to H is nonamenable.

1. INTRODUCTION

A connected graph of bounded degree X is *nonamenable* if X has nonzero Cheeger constant or, equivalently, if the spectral radius of the simple random walk on X is less than one (see Section 2 below for more precise definitions). Nonamenable graphs play an increasingly important role in the study of various probabilistic phenomena, such as random walks, harmonic analysis, Brownian motion, and percolations on graphs and manifolds (see for example [2, 5, 6, 7, 15, 17, 18, 24, 30, 43, 44, 62, 71, 72]), as well as in the study of expander families of finite graphs (see for example [52, 66, 67]).

It is well-known that a finitely generated group G is nonamenable if and only if the Cayley graph of G with respect to some (any) finite generating set is nonamenable. The notion of a *word-hyperbolic group* was introduced by M. Gromov [40] and has played a central role in Geometric Group Theory for the last fifteen years. Word-hyperbolic groups are nonamenable unless they are virtually cyclic. Thus the Cayley graphs of word-hyperbolic groups provide a large and interesting class of nonamenable graphs. In this paper we investigate nonamenability of Schreier coset graphs corresponding to subgroups of hyperbolic groups.

We recall the definition of a Schreier coset graph:

DEFINITION 1.1. Let G be a group and let $\pi: A \to G$ be a map where A is a finite alphabet such that $\pi(A)$ generates G (we refer to such an A as a *marked finite generating set* or just *a finite generating set* of G). Let $H \leq G$ be a subgroup of G. The *Schreier coset graph* (or the *relative Cayley graph*) $\Gamma(G,H,A)$ for G relative to H with respect to A is an oriented labeled graph defined as follows:

1. The vertices of $\Gamma = \Gamma(G,H,A)$ are precisely the cosets of H in G, that is $V\Gamma := \{Hg \mid g \in G\}$.
2. The set of positively oriented edges of $\Gamma(G,H,A)$ is in one-to-one correspondence with the set $V\Gamma \times A$. For each pair $(Hg,a) \in V\Gamma \times A$ there is a positively oriented edge in $\Gamma(G,H,A)$ from Hg to $Hg\pi(a)$ labeled by the letter a.

Thus the label of every path in $\Gamma(G,H,A)$ is a word in the alphabet $A \cup A^{-1}$. The graph $\Gamma(G,H,A)$ is connected since $\pi(A)$ generates G. Moreover, $\Gamma(G,H,A)$ comes equipped with a natural simplicial metric obtained by giving every edge length one.

We can identify the Schreier graph $\Gamma(G,H,A)$ with the 1-skeleton of the covering corresponding to $H \leq G$ of the presentation complex of G based on any presentation of the form $G = \langle A \mid R \rangle$. If M is a closed Riemannian manifold and $H \leq G = \pi_1(M)$, then the Schreier graph $\Gamma(G,H,A)$ is quasi-isometric to the covering space of M corresponding to H. If H is normal in G and $G_1 = G/H$ is the quotient group, then $\Gamma(G,H,A)$ is exactly the Cayley graph of the group G_1 with respect to A. In particular, if $H = 1$ then $\Gamma(G,1,A)$ is the standard *Cayley graph of G with respect to A*, denoted $\Gamma(G,A)$.

A subgroup H of a word-hyperbolic group G is said to be *quasiconvex* in G if for any finite generating set A of G there is $\epsilon \geq 0$ such that every geodesic in $\Gamma(G,A)$ with both endpoints in H is contained in the ϵ-neighborhood of H in $\Gamma(G,A)$. Quasiconvex subgroups are closely related to geometric finiteness in the Kleinian group context [69]. They enjoy a number of particularly good properties and play an important role in hyperbolic group theory and its applications (see for example [3, 4, 8, 31, 34, 35, 36, 37, 38, 42, 45, 46, 48, 51, 53, 55, 61, 70]).

Our main result is the following:

THEOREM 1.2. *Let G be a nonelementary word-hyperbolic group with a marked finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. Then the Schreier coset graph $\Gamma(G,H,A)$ is nonamenable.*

The study of Schreier graphs arises naturally in various generalizations of J. Stallings' theory of ends of groups [23, 29, 60, 61, 63]. The case of virtually cyclic (and hence quasiconvex) subgroups of hyperbolic groups is particularly important to understand in the theory of JSJ-decomposition for hyperbolic groups originally developed by Z. Sela [65] and later by B. Bowditch [11] (see also [59, 23, 28, 64] for various generalizations of the JSJ-theory). A variation of the Følner criterion of nonamenability (see Proposition 2.3 below), when the Cheeger constant is defined by taking the infimum over all finite subsets containing no more than a half of all the vertices, is used to define an important notion of *expander families* of finite graphs. Most known sources of expander families involve taking Schreier coset graphs corresponding to subgroups of finite index in a group with the Kazhdan property (T) (see [52, 66, 67] for a detailed exposition on expander families and their connections with nonamenability).

Since nonamenable graphs of bounded degree are well-known to be *transient* with respect to the simple random walk, Theorem 1.2 implies that $\Gamma(G,H,A)$ is also transient. M. Gromov [40] stated (see R. Foord [27] and I. Kapovich [49] for the proofs) that for any quasiconvex subgroup H in a hyperbolic group G with a finite generating set A, the coset graph $\Gamma(G,H,A)$ is a hyperbolic metric space. A great deal is known about random walks on hyperbolic graphs, but most of these results assume some kind of nonamenability. Thus Theorem 1.2 together with hyperbolicity of $\Gamma(G,H,A)$ and a result of A. Ancona [2] (see also [72]) immediately imply:

COROLLARY 1.3. *Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G and let Y be the Schreier coset graph $\Gamma(G,H,A)$. Then:*

1. *The trajectory of almost every simple random walk on Y converges in the topology of $Y \cup \partial Y$ to some point in ∂Y (where ∂Y is the hyperbolic boundary).*
2. *The Martin boundary of a simple random walk on Y is homeomorphic to the hyperbolic boundary ∂Y, and the Martin compactification of Y corresponding to the simple random walk on Y is homeomorphic to the hyperbolic compactification $Y \cup \partial Y$.*

Let us illustrate Theorem 1.2 for the case of a free group. Let $F = F(a,b)$ be free of rank two and let $H \le F$ be a finitely generated subgroup of infinite index (which is therefore quasiconvex [68]). Set $A = \{a,b\}$. Then the Schreier graph $Y = \Gamma(F,H,A)$ looks like a finite graph with several infinite tree-branches attached to it (the "branches" are 4-regular trees except for the attaching vertices). In this situation it is easy to see that Y has positive Cheeger constant and so Y is nonamenable. Alex Lubotzky and Andrzej Zuk pointed out to the author that if G is a group with the Kazhdan property (T), then for any subgroup H of infinite index in G the Schreier coset graph for G relative to H is nonamenable. There are many examples of word-hyperbolic groups with Kazhdan property (T) (see for instance [73]) and in view of Theorem 1.2 it would be particularly interesting to investigate if they can possess non-quasiconvex finitely generated subgroups.

Nonamenability of graphs is closely related to cogrowth:

COROLLARY 1.4. *Let $G = \langle x_1,\dots,x_k \mid r_1,\dots,r_m\rangle$ be a nonelementary word-hyperbolic group and let $H \le G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1,\dots,x_k\}^{\pm 1}$ of length n representing elements of H. Let b_n be the number of all words in A of length n that represent elements of H. Then*

$$\limsup_{n\to\infty} \sqrt[n]{a_n} < 2k-1$$

and

$$\limsup_{n\to\infty} \sqrt[n]{b_n} < 2k\,.$$

In [10, 50] Theorem 1.2 and Corollary 1.4 play a useful role in obtaining results about "generic-case" complexity of the membership problem as well as about some interesting measures on free groups.

It is easy to see that the statement of Theorem 1.2 need not hold for finitely generated subgroups which are not quasiconvex. For example, a remarkable construction of E. Rips [58] states that for any finitely presented group Q there is a short exact sequence

$$1 \to K \to G \to Q \to 1\,,$$

where G is nonelementary torsion-free word-hyperbolic and where K can be generated by two elements (but K is usually not finitely presentable). If Q is chosen to be infinite amenable, then $[G:K] = \infty$ and the Schreier graph for G relative to K is amenable. Finitely presentable and even hyperbolic examples of such subgroups are also possible. For instance, if F is a free group of finite rank and $\phi\colon F \to F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case $M_\phi/F \simeq \mathbf{Z}$ and hence the Schreier graph for M_ϕ relative to F is amenable.

The author is grateful to Laurent Bartholdi, Philip Bowers, Christophe Pittet and Tatiana Smirnova-Nagnibeda for many helpful discussions regarding random walks, to Pierre de la Harpe and Peter Brinkmann for their careful reading of the paper and numerous valuable suggestions and to Paul Schupp for encouragement.

2. Nonamenability for graphs

Let X be a connected graph of bounded degree. We define the *spectral radius* $\rho(X)$ of X as

$$\rho(X) := \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$$

where x, y are two vertices of X and $p^{(n)}(x,y)$ is the probability that an n-step simple random walk starting at x will end up at y. It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x, y.

Definition 2.1 (Amenability for graphs). A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *nonamenable* if $\rho(X) < 1$.

It is also well-known that nonamenability of X implies that X is *transient*, that is that for a simple random walk on X the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

Convention 2.2. Let X be a connected graph of bounded degree with the simplicial metric d. For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S.