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is the complex number

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}.$$

The group  $GL(q, \mathbf{C}) \simeq \mathbf{C}^*$  acts on the upper halfplane by  $(\lambda, z) \mapsto |\lambda|^2 z$  and so the orbits are described by the argument of the complex number  $z$ . So the characteristic invariant in this case is just

$$\arg \left( (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1} \right).$$

It is equivalent to the invariant  $\theta$  considered in [KR]. This invariant, almost in our terms, was known to E. Cartan (see [Ca]).

REMARK 2. Let us consider the case where  $p = q$ . Then the Stiefel manifold is  $U(q)$ , and the content of Proposition 4.2 is that for  $(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3$

$$T(\sigma_1, \sigma_2, \sigma_3) = (1 - \sigma_2^* \sigma_1)^{-1} (1 - \sigma_2^* \sigma_3) (1 - \sigma_1^* \sigma_3)^{-1}$$

is an invertible skew-Hermitian matrix. The orbits of  $GL(q, \mathbf{C})$  in its action on nondegenerate Hermitian forms are characterized by the signature. So the characteristic invariant as described in Theorem 4.3 in this case reduces to  $\text{sgn } iT(\sigma_1, \sigma_2, \sigma_3)$ . As concerns Theorem 4.4, notice that the invariant  $S$  is trivial (equal to  $-\mathbf{1}_q$ ), so one is only concerned with the invariant  $\arg \det T$ . The bounded domain  $D$  is of tube type and the description of the invariant through the function  $\arg \det$  coincides with the approach of this problem in [CØ], where the invariant was introduced under the name of *generalized Maslov index*.

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