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3. Intersecting chords theorem for convex C^2-domains

Assume that D is a bounded, convex domain in $\mathbf{R}^n$ with C^2-smooth boundary. Let ρ be a C^2-defining function for D, that is, ρ is positive on points in D, negative outside $\bar{D}$ and zero on ∂D. Moreover the gradient $\nabla \rho =: \nu(x)$ is a unit vector field normal to ∂D directed inside D. The *curvature* (or *Weingarten*) *operator* $W_x \colon T_x\partial D \to T_x\partial D$ is by definition the directional derivative of ν in the direction v. The *second fundamental form* is the bilinear form II_x on $T_x\partial D$ given by

$$II_x(v,w) = (w, W_x(v)) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial x_i \partial x_j} v_i w_j \, .$$

The value $II_x(u,u) =: k_x(u)$ is called *the normal curvature* of ∂D at x in the direction of the unit tangent vector u. We will assume that the curvature of ∂D is everywhere nonzero, meaning that II is everywhere positive definite, so there is a constant $k_D > 0$ such that

$$k_D^{-1} \le k_x(u) \le k_D \tag{3.1}$$

for every $u \in UT_x\partial D$ and $x \in \partial D$.

In this section we will establish:

THEOREM 3.1. *Let D be a bounded convex domain in $\mathbf{R}^n$. Suppose that the boundary ∂D is smooth of class C^2 and the curvature of ∂D is everywhere nonzero. Then there is a constant $C > 0$ such that*

$$C^{-1} \le \frac{K(x,y,z)}{K(x',y',z')} \le C$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane.

In view of Corollary 1.2 and Theorem 2.1 this implies:

COROLLARY 3.2. *Let D be as above. Then D has the intersecting chords property and (D,h) is Gromov hyperbolic.*

3.1 The two-dimensional case

For this subsection, let D be a convex, bounded domain in $\mathbf{R}^2$ with C^2-boundary curve ∂D. Assume in addition that the differential geometric curvature κ is positive (nonzero) at every point of ∂D.

LEMMA 3.3. *The Menger curvature $K(x,y,z)$ of three points extends to a continuous function on $\partial D \times \partial D \times \partial D$. The value $K(x,x,z)$ equals the curvature of the circle tangent to ∂D at x and passing through z, and the value $K(x,x,x)$ equals $\kappa(x)$.*

Proof. The continuity for three distinct points is clear. When three points converge to one point on the boundary, it is a standard fact that K converges to κ, see [Sp78, Ch. 1], or [BG88, p. 304 or p. 306]. When y_t converges to $x \neq z$, then $d(y_t, z) \to d(x,z)$ and $\sin \angle(y_t x, xz) \to \sin \angle(T_x \partial D, xz)$. This proves the continuity and it is clear that the limit circle is tangent to ∂D at x. □

The idea of the proof of the following proposition was supplied to us by M. Bucher.

PROPOSITION 3.4. *Let (x,y,z) be a global minimum or maximum point for K on $\partial D \times \partial D \times \partial D$. Then ∂D contains the shortest circle arc connecting x, y and z.*

Proof. Recall the formula (1.2) and consider the circle in question through the three boundary points x, y, z with extremal, say maximal, radius. Denote by γ a shortest arc on this circle connecting these three points, and assume that x and z are the boundary points of γ.

In the case $x = y = z$ there is nothing to prove. Assume now that the three points are all distinct and consider first a potential boundary point w between γ and xz. By convexity of D it cannot lie inside the triangle xyz.

If γ is larger than a halfcircle, then note that (depending on which region w belongs to) either $R(x,w,y) > R(x,y,z)$ or $R(z,w,y) > R(x,y,z)$ (compare the angle at w with the one at either z or x). Therefore w cannot belong to ∂D. If γ is less than a half-circle, then, again by looking at the angles and using the formula for R, we have $R(x,w,z) > R(x,y,z)$, for any such w.

Secondly, note that a potential boundary point w outside the circle in the half-plane defined by the line through x and z containing y cannot belong to ∂D, because either $R(w,y,z)$ or $R(x,y,w)$ (depending on where w lies) is greater than $R(x,y,z)$. Hence the arc γ must coincide with an arc of ∂D.

In the case $x = y \neq z$, no point outside the circle can lie on ∂D, again by the assumption on the maximality of the radius. On the other hand, a point w between γ and xz cannot belong to ∂D because $R(x,w,z) > R(x,x,z)$, and again we have the desired conclusion.

The case of maximal curvature can be treated analogously. □

In view of the continuity of κ, the following immediate consequence of Proposition 3.4 is somewhat analogous to a mean value theorem.

COROLLARY 3.5. *Denote by* κ_{min} *and* κ_{max} *the minimum and the maximum, respectively, of the curvature of* ∂D. *Then*

$$\kappa_{min} \leq K(x,y,z) \leq \kappa_{max} \,,$$

for any three boundary points x, y, z.

3.2 THE PROOF OF THEOREM 3.1

Assume that D is as in the theorem. To simplify the notation we will only discuss the 3-dimensional case. Each 2-dimensional plane section is Gromov hyperbolic by the above so we only need an overall bound for constants $\delta(S)$ when S runs through all the plane sections. The intersection of ∂D with a 2-dimensional plane gives rise to a smooth planar curve α, which we assume is parameterized by arclength. The constant δ of the hyperbolicity depends on the curvature of α. These curves could have an arbitrarily large curvature but we need only to bound from above (and hence from below) the ratio of the curvatures at different points of the curve. The curvature vector $\alpha''(t)$ of α at a point $x = \alpha(t)$ lies in this plane and is orthogonal to $\alpha'(t)$. Thus we need to bound the ratio $\frac{|\alpha''(t)|}{|\alpha''(s)|}$. It is a fact (Meusnier's lemma, see [Kl78, p. 43] that

$$k_x(\alpha'(t)) = |\alpha''(t)| \cos\theta(t) \,,$$

where $k_x(\alpha'(t)) = II_x(\alpha'(t), \alpha'(t))$ is the normal curvature in the direction $\alpha'(t)$ and $\theta(t)$ is the angle between $\alpha''(t)$ and the normal of ∂D at x. In view of the assumption (3.1) and Corollary 3.5 we therefore need to bound the ratio $\frac{\cos\theta(s)}{\cos\theta(t)}$ independently of s, t and α. Near any point x the surface ∂D is the graph of a C^2 function $z = f(x,y)$ in suitable Cartesian coordinates. Hence any small plane section C_ε is given by the equation $f(x,y) = \varepsilon > 0$. Expressing θ in terms of f we arrive at the problem of bounding the ratio of the gradients $\frac{|\nabla f(p)|}{|\nabla f(q)|}$ along the section. By rotation in the xy-plane we may assume that the x- and y-axis are along the direction of principal curvature. By developing $f(x,y)$ into a Taylor's expansion around the origin, we obtain $f(x,y) = \frac{1}{2}(ax^2 + by^2) + r$, where r vanishes at $(0,0)$ together with all its derivatives up to second order, and where $a = f_{xx}(0,0), b = f_{yy}(0,0)$ are the principal curvatures. We conclude that $c < \frac{|\nabla f(x,y)|}{\sqrt{x^2+y^2}} < C$ near 0 for universal $c, C > 0$ and thus it remains to bound the ratio $\frac{x^2+y^2}{x'^2+y'^2}$ on C_ε. But this ratio