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Autor: GROMOV, Mikhaïl
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me to write this paper. I also appreciate the hospitality of IHES, which made possible my involvement in this story. I am especially thankful to Dennis Sullivan, who took pains to read the paper and to clean it up of multiple errors.

STRUCTURE OF THE PAPER. We start with a geometric outlook on topological entropy and reduce our inequality (0.1) to standard facts about minimal varieties. We discuss next the real algebraic analogue of (0.1) and a generalization to maps. We conclude with an estimate of the entropy involving the mean curvature.

§1. NOTATION AND DEFINITIONS

For a space X we denote by X^k the product $X \times X \times \dots \times X$ (k factors). A *graph* Γ over X is by definition an arbitrary set $\Gamma \subset X^2$. When X is finite this is the usual definition of an oriented graph (with loops). The graph of a map $X \rightarrow X$ gives another example.

For a graph Γ we denote by $\Gamma_k \subset X^k$ the set of strings $(x_1, \dots, x_i, \dots, x_k)$, $x_i \in X$, where each pair $(x_{i-1}, x_i) \in X^2$ is contained in Γ .

When X is endowed with a metric, we call ϵ -cubes products in X^k of balls from X of radius ϵ . For a set $Y \subset X^k$ we denote by $\text{Cap}_\epsilon Y$ the minimal number of ϵ -cubes needed to cover Y .

ENTROPY

Set $h_\epsilon(\Gamma) = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \text{Cap}_\epsilon \Gamma_k$, and $h(\Gamma) = \lim_{\epsilon \rightarrow 0} h_\epsilon(\Gamma)$, for $\Gamma \subset X^2$.

When f is an endomorphism $X \rightarrow X$, we define its *entropy* $h(f)$ as the entropy of its graph Γ_f . If the space X is compact, the definition does not depend on the choice of the metric [2]. Observe that the entropy of a general graph Γ is equal to the entropy of the shift in $\Gamma_\infty \subset X^\infty$: Γ_∞ is the space of doubly infinite strings $(x_i)_{i=\dots, -1, 0, 1, \dots}$ with the product topology, and the shift maps (x_i) to (x_{i+1}) . For finite X , we come to the usual definition of the Markov shift.

VOLUME

From now on, X is a Riemannian manifold and $n = \dim X$, $\Gamma \subset X^2$. We denote by $\text{Vol} \Gamma_k$ the n -dimensional volume of $\Gamma_k \subset X^k$, i.e. the

n -dimensional Hausdorff measure with respect to the Riemann product metric in X^k . Set

$$\text{lov } \Gamma = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \text{Vol } \Gamma_k.$$

For an f we set $\text{lov } f = \text{lov } \Gamma_f$. This is a smooth invariant of f (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic f we are going to prove that

$$(1.0) \quad h(f) \leq \text{lov } f.$$

DENSITY

Denote by $\text{Dens}_\epsilon(\Gamma_k, \gamma)$, for $\gamma \in \Gamma_k \subset X^k$, the n -dimensional measure of the intersection of Γ_k with the ball (in the Riemannian product metric) of radius ϵ centered at γ . Set $\text{Dens}_\epsilon(\Gamma_k) = \inf_{\gamma \in \Gamma_k} \text{Dens}_\epsilon(\Gamma_k, \gamma)$, and then $\text{lodn}_\epsilon \Gamma = \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{Dens}_\epsilon \Gamma_k$, and finally

$$\text{lodn } \Gamma = \lim_{\epsilon \rightarrow 0} \text{lodn}_\epsilon \Gamma.$$

Observe that $\text{Vol} \geq \text{Cap}_{2\epsilon} \text{Dens}_\epsilon$ and hence that

$$(1.1) \quad h(V) \leq \text{lov } \Gamma - \text{lodn } \Gamma.$$

§2. ESTIMATES OF DENSITY

Consider a Riemannian manifold W (it will be X^k in the sequel) and an n -dimensional subvariety $V \subset W$. We suppose that the boundary of V (if there is such) does not intersect the ball $B_\epsilon \subset W$ of radius $\epsilon > 0$ centered at a point $v_0 \in V$. We suppose also that the injectivity radius of W at v_0 is at least ϵ , i.e. the exponential map $T_{v_0}(W) \rightarrow W$ is injective in the ϵ -ball in $T_{v_0}(W)$.

DENSITY OF A MINIMAL VARIETY

If the sectional curvature in B_ϵ is not greater than K and V is minimal, then

$$(2.0) \quad \text{Vol}(V \cap B_\epsilon) \geq C > 0,$$

where the constant C depends on n , K , and ϵ , but does not depend on $\dim W$.