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Kapitel: §1. Notation and definitions

DOI: https://doi.org/10.5169/seals-66687 me to write this paper. I also appreciate the hospitality of IHES, which made possible my involvement in this story. I am especially thankful to Dennis Sullivan, who took pains to read the paper and to clean it up of multiple errors.

STRUCTURE OF THE PAPER. We start with a geometric outlook on topological entropy and reduce our inequality (0.1) to standard facts about minimal varieties. We discuss next the real algebraic analogue of (0.1) and a generalization to maps. We conclude with an estimate of the entropy involving the mean curvature.

§1. NOTATION AND DEFINITIONS

For a space X we denote by X^k the product $X \times X \times \dots \times X$ (k factors). A *graph* Γ over X is by definition an arbitrary set $\Gamma \subset X^2$. When X is finite this is the usual definition of an oriented graph (with loops). The graph of a map $X \to X$ gives another example.

For a graph Γ we denote by $\Gamma_k \subset X^k$ the set of strings $(x_1, \dots, x_i, \dots, x_k)$, $x_i \in X$, where each pair $(x_{i-1}, x_i) \in X^2$ is contained in Γ.

When X is endowed with a metric, we call ϵ-*cubes* products in X^k of balls from X of radius ϵ. For a set $Y \subset X^k$ we denote by $\operatorname{Cap}_\epsilon Y$ the minimal number of ϵ-cubes needed to cover Y.

ENTROPY

Set $h_\epsilon(\Gamma) = \limsup_{k\to\infty} \frac{1}{k} \log \operatorname{Cap}_\epsilon \Gamma_k$, and $h(\Gamma) = \lim_{\epsilon\to 0} h_\epsilon(\Gamma)$, for $\Gamma \subset X^2$.

When f is an endomorphism $X \to X$, we define its *entropy* $h(f)$ as the entropy of its graph Γ_f. If the space X is compact, the definition does not depend on the choice of the metric [2]. Observe that the entropy of a general graph Γ is equal to the entropy of the shift in $\Gamma_\infty \subset X^\infty$: Γ_∞ is the space of doubly infinite strings $(x_i)_{i=\dots,-1,0,1,\dots}$ with the product topology, and the shift maps (x_i) to (x_{i+1}). For finite X, we come to the usual definition of the Markov shift.

VOLUME

From now on, X is a Riemannian manifold and $n = \dim \Gamma$, $\Gamma \subset X^2$. We denote by $\operatorname{Vol} \Gamma_k$ the n-dimensional volume of $\Gamma_k \subset X^k$, i.e. the

n-dimensional Hausdorff measure with respect to the Riemann product metric in X^k. Set

$$\operatorname{lov} \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log \operatorname{Vol} \Gamma_k .$$

For an f we set $\operatorname{lov} f = \operatorname{lov} \Gamma_f$. This is a smooth invariant of f (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic f we are going to prove that

$$h(f) \leq \operatorname{lov} f . \tag{1.0}$$

Density

Denote by $\operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, for $\gamma \in \Gamma_k \subset X^k$, the n-dimensional measure of the intersection of Γ_k with the ball (in the Riemannian product metric) of radius ϵ centered at γ. Set $\operatorname{Dens}_\epsilon(\Gamma_k) = \inf_{k\in\Gamma_k} \operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, and then $\operatorname{lodn}_\epsilon \Gamma = \liminf_{k\to\infty} \frac{1}{k} \log \operatorname{Dens}_\epsilon \Gamma_k$, and finally

$$\operatorname{lodn} \Gamma = \lim_{\epsilon\to 0} \operatorname{lodn}_\epsilon \Gamma .$$

Observe that $\operatorname{Vol} \geq \operatorname{Cap}_{2\epsilon} \operatorname{Dens}_\epsilon$ and hence that

$$h(V) \leq \operatorname{lov} \Gamma - \operatorname{lodn} \Gamma . \tag{1.1}$$

§2. Estimates of density

Consider a Riemannian manifold W (it will be X^k in the sequel) and an n-dimensional subvariety $V \subset W$. We suppose that the boundary of V (if there is such) does not intersect the ball $B_\epsilon \subset W$ of radius $\epsilon > 0$ centered at a point $v_0 \in V$. We suppose also that the injectivity radius of W at v_0 is at least ϵ, i.e. the exponential map $T_{v_0}(W) \to W$ is injective in the ϵ-ball in $T_{v_0}(W)$.

Density of a minimal variety

If the sectional curvature in B_ϵ is not greater than K and V is minimal, then

$$\operatorname{Vol}(V \cap B_\epsilon) \geq C > 0 , \tag{2.0}$$

where the constant C depends on n, K, and ϵ, but does not depend on $\dim W$.