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APPENDIX: EXAMPLES OF HOLOMORPHIC ENDOMORPHISMS

The following examples appeared after the discussion I had with Spencer Bloch and David Mumford.

(1) TWISTED HOPF MANIFOLDS

Divide $\mathbf{C}^n \setminus 0$ by the action of a linear operator A without eigenvalues in the unit disk. All endomorphisms of the quotient ($n > 1$) come from polynomial maps $\tilde{f}: \mathbf{C}^n \rightarrow \mathbf{C}^n$. For such an endomorphism, its entropy and "lov" are probably equal to "log deg".

EXAMPLE. $A: (z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2)$ and $\tilde{f}: (z_1, z_2) \mapsto (z_1^p, z_2^p)$.

(2) GENERALIZED HOPF MANIFOLDS

Let $f_0: X_0 \rightarrow X_0$ be an endomorphism. Take a line bundle L over X_0 such that $f_0^*(L) = \bigotimes^p L$ ($:= L \otimes \cdots \otimes L$, p times). Locating such an L is usually quite easy by looking at $\text{Pic}(X_0)$. Denote by Y the total space of L . There is a fiberwise map $\tilde{f}: Y \rightarrow Y$ lifting f_0 and acting on fibers as $z \mapsto z^p$. If we divide Y by a fiberwise action of \mathbf{Z} (it is $z \mapsto z_0 z$, $z_0 \neq 0$, in each fiber) we get $f: Y/\mathbf{Z} \rightarrow Y/\mathbf{Z}$.

There is another way to compactify Y by taking the total space of the one-dimensional projective bundle associated to L . The endomorphism \tilde{f} canonically extends to this compactification.

(3) THE CALABI-ECKMANN MANIFOLDS

Let us take $(\mathbf{C}^k \times \mathbf{C}^\ell) \setminus ((\mathbf{C}^k \times 0) \cup (0 \times \mathbf{C}^\ell))$ and divide by the following action of \mathbf{C} :

$$(z_1, z_2) \mapsto (A_1^\lambda z_1, A_2^\lambda z_2), \quad \lambda \in \mathbf{C}.$$

A_1 and A_2 are appropriate linear operators in \mathbf{C}^k and \mathbf{C}^ℓ . For example, $A_1^\lambda = \exp \lambda$, $A_2^\lambda = \exp i\lambda$, where λ is a scalar. In the last case, the factor manifold possesses an endomorphism f which lifts to the following polynomial map

$$\mathbf{C}^k \times \mathbf{C}^\ell \rightarrow \mathbf{C}^k \times \mathbf{C}^\ell : (z_1, \dots, z_{k+\ell}) \mapsto (z_1^p, \dots, z_{k+\ell}^p).$$

Recall that the Calabi-Eckmann manifolds are diffeomorphic to $\mathbf{S}^{2k-1} \times \mathbf{S}^{2\ell-1}$. The above map f has degree $d = (2(k + \ell - 1))^p$ and $h(f) = \text{lov } f = \log d$.

(4) BLOWING UP

Let us take $W \subset V_0$ and an endomorphism $f: V_0 \rightarrow V_0$ such that $f^{-1}(W) = W$. The endomorphism f can be sometimes lifted to the manifold V obtained by blowing up W .

EXAMPLE. $V_0 = \mathbf{CP}^1 \times \mathbf{CP}^1$, W is the single point $(0, 0)$, and $f: (z_1, z_2) \mapsto (z_1^p, z_2^p)$.

(5) CONCLUDING REMARKS

A typical compact complex manifold has very few endomorphisms. For example, manifolds with nontrivial Kobayashi volume have no endomorphisms of degree ≥ 2 . Do Grassmann manifolds have such endomorphisms? (No, see [3'].)

REFERENCES

- [1] ARTIN, M. and B. MAZUR. On periodic points. *Ann. of Math. (2)* 81 (1965), 82–99.
- [1'] KALOSHIN, V. YU. and B. R. HUNT. A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms I. *Electronic Research Announcements of AMS* 7 (April 18, 2001), 17–27.
- [2] BOWEN, R. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* 153 (1971), 401–414.
- [3] MILNOR, J. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.* 15 (1964), 275–280.
- [3'] PARANJAPPE, K. H. and V. SRINIVAS. Self maps of homogeneous spaces. *Invent. Math.* 98 (1989), 425–444.
- [4] MISIUREWICZ, M. and F. PRZYTYCKI. Topological entropy and degree of smooth mappings. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 25 (1977), 573–574.
- [5] SHUB, M. Dynamical systems, filtrations and entropy. *Bull. Amer. Math. Soc.* 80 (1974), 27–41.
- [5'] GROMOV, M. *Partial Differential Relations*. Springer, 1986.

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