**Zeitschrift:** L'Enseignement Mathématique
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique
**Band:** 49 (2003)
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HYPERBOLICITY OF MAPPING-TORUS GROUPS AND SPACES
**Autor:** Gautero, François
**Kapitel:** 7. Substitution of quasi geodesics
**DOI:** https://doi.org/10.5169/seals-66690

Hence $|I_C|_{f(I)} \geq \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}} |I_D|_{f(I)}$, so that $|I_C|_{f(I)} \geq \frac{X(n)}{1+X(n)} |I|_{f(I)}$ with $X(n) = \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}}$. Now $\lim_{n \to +\infty} \frac{X(n)}{1+X(n)} = 1$, so that for some $n_* \geq 1$, for any $n \geq n_*$, $\frac{X(n)}{1+X(n)} \geq \frac{1}{2}$. Since the horizontal length of any interval $I_k$ in $I_C$ is at most $C_{6.2}(J,J')$, and the telescopic length of the associated $p_k \subset p$ is at least $t_0$, we obtain

$$|p|_{(\widetilde{X},\mathcal{H})} \geq \frac{t_0}{2C_{6.2}(J,J')} |I|_{f(I)}.$$

On the other hand, $|p|_{(\widetilde{X},\mathcal{H})} \leq 2Jnt_0 + \lambda^{-n} J |I|_{f(I)} + J'$ for any $n \geq n_*$. The last two inequalities give, for $n \geq n_*$, $2Jnt_0 + \lambda^{-n} J |I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J,J')} |I|_{f(I)}$, equivalently $2Jnt_0 + J' \geq (\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n} J) |I|_{f(I)}$. We choose $n_\circ \geq n_*$ such that $\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J > 0$. We get

$$\frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J} \geq |I|_{f(I)}.$$

Thus, for $|I|_{f(I)} > \frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J}$, $h$ is not dilated in the future after $t_0$. If $|I|_{f(I)} > \lambda_+^{n_\circ} M$, then $|h|_{f(h)} \geq M$. Therefore $h$ is dilated in the past after $t_0$. We choose $N$ such that $\lambda^N \lambda_+^{-n_\circ} > \lambda$. Thus, if $|I|_{f(I)} \geq \max(\lambda_+^{n_\circ} M, \frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ} J})$ then $I$ is dilated in the past after $(n_\circ C_{6.2}(J,J') + N) t_0$. The arguments and computations in the case where $\max_{x \in p} f(x) \leq f(I)$ are the same. □

## 7. Substitution of quasi geodesics

LEMMA 7.1. *Let $p$ be a $(J,J')$-quasi geodesic. Let $q$ be obtained from $p$ by replacing subpaths $p_i \subset p$ by $(L,L')$-quasi geodesics $q_i$ satisfying the following properties:*

- *$q_i$ has the same endpoints as $p_i$,*
- *$q_i$ is $L$-close to $p_i$,*
- $|q_i|_{(\widetilde{X},\mathcal{H})} \leq L |p_i|_{(\widetilde{X},\mathcal{H})}$.

*There exists a constant $C_{7.1}(L,L',J,J')$, which increases in each variable, such that $q$ is a $(C_{7.1}(L,L',J,J'), C_{7.1}(L,L',J,J'))$-quasi geodesic which is $L$-close to $p$.*

*Proof.* Since each $q_i$ is $L$-close to a $p_i$, and with the same endpoints, $q$ is $L$-close to $p$. Let us consider any two points $x$, $y$ in $q$ and let $q_{xy} \subset q$

be the subpath of $q$ between $x$ and $y$. If both $x$ and $y$ lie in a $q_i$, or in a same subpath in the closed complement of the union of the $q_i$'s, then $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq \max(L,J)d_{(\widetilde{X},\mathcal{H})}(x,y) + \max(L',J')$. Otherwise $q_{xy} = w_1w_2w_3$, where $w_1$, $w_3$ are contained either in some $q_i$ or in $p$, and $w_2$ begins and ends with the initial or terminal point of some $q_i$. The third property concerning the $q_i$'s leads to $|w_2|_{(\widetilde{X},\mathcal{H})} \leq L|p_2|_{(\widetilde{X},\mathcal{H})}$, where $p_2 \subset p$ is the subpath of $p$ with the same endpoints as $w_2$. Thus $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq LJd_{(\widetilde{X},\mathcal{H})}(x,y) + 2\max(L',LJ')$. □

LEMMA 7.2. *Let $p$ be a straight $(J,J')$-quasi geodesic $-$-hole such that $\max_{x\in p} f(I)-f(x) \leq L$, where $I$ is the horizontal geodesic joining the endpoints of $p$. Then there exists a constant $C_{7.2}(L,J,J') \geq M$, which increases in each variable, such that*

1) $|I|_{f(I)} \leq C_{7.2}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

2) *$I$ is a straight $(C_{7.2}(L,J,J'), C_{7.2}(L,J,J'))$-quasi geodesic which is $C_{7.2}(L,J,J')$-close to $p$.*

*Proof.* A horizontal geodesic is always straight. The horizontal geodesic $I$ is the pulled-tight projection of $p$. Thus, by the bounded-dilatation property, $|I|_{f(I)} \leq \lambda_+^L|p|_{(\widetilde{X},\mathcal{H})}$. By Lemma 5.6, $I$ is $C_{5.6}(L)$-close to $p$. Consider any subpath $I'$ of $I$; it is the pulled-tight projection of some subpath $p'$ of $p$. By the bounded-dilatation property, $|I'|_{f(I)} \leq \lambda_+^L|p'|_{(\widetilde{X},\mathcal{H})}$. Since $p$ is a $(J,J')$-quasi geodesic, $|I'|_{f(I)} \leq \lambda_+^L(Jd_{(\widetilde{X},\mathcal{H})}(i(p'),t(p'))+J')$. Since $I'$ is $C_{5.6}(L)$-close to $p'$, $|I'|_{f(I)} \leq \lambda_+^L Jd_{(\widetilde{X},\mathcal{H})}(i(I'),t(I')) + \lambda_+^L(2JC_{5.6}(L) + J')$. □

LEMMA 7.3. *Let $p$ be a straight $(J,J')$-quasi geodesic $-$-hole such that the horizontal length of the horizontal geodesic $I$ between its endpoints is less than or equal to $L$. Then there exists a constant $C_{7.3}(L,J,J') \geq M$, which increases in each variable, such that*

1) $|I|_{f(I)} \leq C_{7.3}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

2) *$I$ is a straight $(C_{7.3}(L,J,J'), C_{7.3}(L,J,J'))$-quasi geodesic which is $C_{7.3}(L,J,J')$-close to $p$.*

*Proof.* Since $p$ is a $(J,J')$-quasi geodesic,

$$\max_{x\in p}|f(x) - f(I)| \leq J|I|_{f(I)} + J'.$$

Lemma 7.3 now follows from Lemma 7.2. □

LEMMA 7.4. *Let $p$ be a straight $(J,J')$-quasi geodesic stair. For any $L \geq 0$, there exists a constant $C_{7.4}(L,J,J')$, which increases in each variable, such that if $q$ is a straight stair whose points are at horizontal distance at most $L$ from $p$, and with the same endpoints as $p$, then*

1) *$q$ is a straight $(C_{7.4}(L,J,J'), C_{7.4}(L,J,J'))$-quasi geodesic stair which is $L$-close to $p$.*

2) $|q|_{(\widetilde{X},\mathcal{H})} \leq C_{7.4}(L,J,J')|p|_{(\widetilde{X},\mathcal{H})}$.

*Proof.* Consider a stair $S$, in the disc bounded by $p \cup q$, whose endpoints are those of $p$ and $q$, and whose vertical geodesics end at $q$, all the stairs being oriented so that $f$ is increasing along them. Consider a subpath $S'$ of $S$ which is the concatenation of a vertical segment followed by a horizontal one. By assumption, the horizontal length $X$ of $S'$ is bounded above by $L$. Let $t$ be its vertical length. The bounded-dilatation property implies that the quotient of $|S'|_{(\widetilde{X},\mathcal{H})}$ by the telescopic length of the subpath of $p$ between the endpoints of $S'$ is bounded above by $Q = \frac{t+X}{t+\lambda_+^{-t}X}$. Since $X \leq L$, $Q$ tends to $1$ as $t \to +\infty$. One thus obtains a constant $T$ such that for $t \geq T$, $Q$ is bounded above by some constant, depending on $L$. When both $t$ and $X$ are close to $0$ then $Q$ is close to $1$. Hence, since $Q$ is continuous, $Q$ admits an upper bound, denoted by $A(L)$, for all the $t$ and $X$ considered. This upper bound will be the same for all the subpaths $S'$ as above.

The stair $S$ is a concatenation of such subpaths $S'$, possibly with one or two subpaths of $p$ at the extremities. Thus the additivity of the telescopic length gives $|S|_{(\widetilde{X},\mathcal{H})} \leq A(L)|p|_{(\widetilde{X},\mathcal{H})}$. Let $S''$ be a subpath of $S$ which is the concatenation of a horizontal subpath followed by a vertical one. The path $S$ is the concatenation of such subpaths $S''$ with possibly one or two subpaths of $q$ at the extremities. Exactly the same arguments as above give $|q|_{(\widetilde{X},\mathcal{H})} \leq A(L)|S|_{(\widetilde{X},\mathcal{H})}$. We thus get $|q|_{(\widetilde{X},\mathcal{H})} \leq A(L)^2|p|_{(\widetilde{X},\mathcal{H})}$. It only remains to prove that $q$ is a quasi geodesic with constants of quasi geodesicity depending only on $L$, $J$, $J'$. Let $x, y$ be any two points in $q$. As usual $q_{xy}$ is the subpath of $q$ between $x$ and $y$ and we denote by $p_{x'y'}$ the subpath of $p$ between the two points $x'$, $y'$ in $p$ which are at horizontal distance at most $L$ from $x$ and $y$. We consider a stair $S$ between $q_{xy}$ and $p_{x'y'}$, with the same endpoints as $q_{xy}$. The same arguments as above apply and give $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq A(L)^2|p_{x'y'}|_{(\widetilde{X},\mathcal{H})}$. Since $p$ is a $(J,J')$-quasi geodesic, we conclude that $|q_{xy}|_{(\widetilde{X},\mathcal{H})} \leq JA(L)^2 d_{(\widetilde{X},\mathcal{H})}(x',y') + J'A(L)^2$. Since $d_{(\widetilde{X},\mathcal{H})}(x',y') \leq d_{(\widetilde{X},\mathcal{H})}(x,y) + 2L$, the proof of Lemma 7.4 is complete. □