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**Autor:** Gautero, François  
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Proposition 10.1 provides a  $\kappa(r, s) = Bi(A(r, s), A(r, s))$  such that this bigon is  $\kappa(r, s)$ -thin. Thus the given  $(r, s)$ -chain bigon is  $\delta(r, s)$ -thin, with  $\delta(r, s) = \kappa(r, s) + 2A(r, s)$ . By Lemma 11.1, the given forest-stack, which is a  $(1, 2)$ -quasi geodesic metric space, is  $2\delta(1, 6)$ -hyperbolic.  $\square$

## 12. BACK TO MAPPING-TELESCOPES

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

### 12.1 STATEMENT OF THE THEOREM

An **R-tree** (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an **R-tree** is a topological tree. An **R-forest** is a union of disjoint **R-trees**.

LEMMA 12.1. *Let  $(\Gamma, d_\Gamma)$  be an **R-forest** and let  $\psi: \Gamma \rightarrow \Gamma$  be a forest-map of  $\Gamma$ . Let  $(K_\psi, f, \sigma_t)$  be the mapping-telescope of  $(\psi, \Gamma)$  equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric  $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$  on  $K_\psi$  such that*

1. *The **R-forests**  $(f^{-1}(r), m_r)$  and  $(f^{-1}(r+1), m_{r+1})$  are isometric. Each stratum  $(f^{-1}(n), m_n)$ ,  $n \in \mathbf{Z}$ , is isometric to  $(\Gamma, d_\Gamma)$ .*
2. *For any real  $r$  and any horizontal geodesic  $g \in f^{-1}(r)$ , the map*

$$l_{r,g}: \begin{cases} ]-1-r] \rightarrow \mathbf{R}^+ \\ t \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

*is monotone.*

*Such a horizontal metric is called a horizontal  $d_\Gamma$ -metric. The telescopic metric associated to a horizontal  $d_\Gamma$ -metric is called a mapping-telescope  $d_\Gamma$ -metric.*

*Proof.* We make each  $\Gamma \times \{n\}$ ,  $n \in \mathbf{Z}$ , an **R-forest** isometric to  $\Gamma$ . We consider a cover of  $\Gamma$  by geodesics of length 1 which intersect only at their endpoints. Each  $\Gamma \times \{n\}$  inherits the same cover. There is a disc  $D_{e,n}$  in  $K_\psi$  for each such horizontal geodesic  $e$  in  $\Gamma \times \{n\}$ . This disc is bounded by  $e$ ,  $\psi(e)$  and the orbit-segments between the endpoints of  $e$  and those of  $\psi(e)$ .

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of  $e$  to that of  $\psi(e)$ . We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum  $f^{-1}(n)$ ,  $n \in \mathbf{Z}$ , is isometric to  $(\Gamma, d_\Gamma)$ . And the maps denoted by  $l_{r,g}$  in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs  $D_{e,n}$  between  $\Gamma \times \{n\}$  and  $\Gamma \times \{n+1\}$  are copies of the discs  $D_{e,n'}$  between  $\Gamma \times \{n'\}$  and  $\Gamma \times \{n'+1\}$ , for any  $n, n'$  in  $\mathbf{Z}$ . This allows us to choose the horizontal metric to satisfy the further condition that  $(f^{-1}(r), m_r)$  be isometric with  $(f^{-1}(r+1), m_{r+1})$  for any real number  $r$ .  $\square$

We now define dynamical properties for **R**-forest maps.

DEFINITION 12.2. Let  $(\Gamma, d_\Gamma)$  be an **R**-forest. A forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *weakly bi-Lipschitz* if there exist  $\mu \geq 1$  and  $K \geq 0$  such that  $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$ .

DEFINITION 12.3. Let  $(\Gamma, d_\Gamma)$  be an **R**-forest. A forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *hyperbolic* if it is weakly bi-Lipschitz and there exist  $\lambda > 1$ ,  $N \geq 1$ ,  $M \geq 0$  such that for any pair of points  $x, y$  in  $\Gamma$  with  $d_\Gamma(x, y) \geq M$ , either  $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$  or  $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$  for some  $x_N, y_N$  with  $\psi^N(x_N) = x$ ,  $\psi^N(y_N) = y$ .

A hyperbolic forest-map  $\psi$  of  $(\Gamma, d_\Gamma)$  is *strongly hyperbolic* if, for any pair of points  $x, y$  with  $d_\Gamma(x, y) \geq M$  and each connected component containing both a preimage of  $x$  and a preimage of  $y$  under  $\psi^N$ , there is at least one pair of such preimages  $x_N, y_N$  for which  $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$ .

If the forest  $\Gamma$  is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. *Let  $(\Gamma, d_\Gamma)$  be an **R**-forest. Let  $\psi$  be a strongly hyperbolic forest-map of  $(\Gamma, d_\Gamma)$  whose mapping-telescope  $K_\psi$  is connected. Then  $K_\psi$  is a Gromov-hyperbolic metric space for any mapping-telescope  $d_\Gamma$ -metric.*