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THE BASIC GERBE OVER A COMPACT SIMPLE LIE GROUP

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ABSTRACT. Let G be a compact, simply connected simple Lie group. We give a construction of an equivariant gerbe with connection on G, with equivariant 3-curvature representing a generator of $H_G^3(G,\mathbf{Z})$. Among the technical tools developed in this context is a gluing construction for equivariant bundle gerbes.

1. INTRODUCTION

Let G be a compact, simply connected simple Lie group, acting on itself by conjugation. It is well-known that the cohomology of G, and also its equivariant cohomology, is trivial in degree less than three and that $H^3(G,\mathbf{Z})$ and $H_G^3(G,\mathbf{Z})$ are canonically isomorphic to $\mathbf{Z}$. The generator of $H^3(G,\mathbf{Z})$ is represented by a unique bi-invariant differential form $\eta \in \Omega^3(G)$, admitting an equivariantly closed extension $\eta_G \in \Omega_G^3(G)$ in the complex of equivariant differential forms. Our goal in this paper is to give an explicit, finite-dimensional description of an equivariant gerbe over G, with equivariant 3-curvature η_G.

A number of constructions of gerbes over compact Lie groups can be found in the literature, using different models of gerbes and valid in various degrees of generality. The differential geometry of gerbes was initiated by Brylinski's book [8], building on earlier work of Giraud. In this framework gerbes are viewed as sheafs of groupoids satisfying certain axioms. Brylinski gives a general construction of a gerbe with connection, for any integral closed 3-form on any 2-connected manifold M. The argument uses the path fibration $P_0M \to M$, and is similar to the well-known construction of a line bundle with connection out of a given integral closed 2-form on a simply connected manifold. In a later paper [9], Brylinski gives a finite-dimensional description of the sheaf of groupoids defining the basic gerbe for any compact Lie group G.

A less abstract picture, developed by Chatterjee-Hitchin [10, 18, 19], describes gerbes in terms of *transition line bundles* similar to the presentation of line bundles in terms of transition functions. A detailed construction of transition line bundles for the basic gerbe over $G = \mathrm{SU}(N)$ (as well as for the much more complicated case of finite quotients of $G = \mathrm{SU}(N)$) was obtained by Gawędzki-Reis [13].

In this paper, we will extend the Gawędzki-Reis approach from $\mathrm{SU}(N)$ to other simply connected simple Lie groups G. A fundamental difficulty in the more general case is that, in contrast to the case $G = \mathrm{SU}(N)$, the pull-back of a generator of $H^3_G(G, \mathbf{Z})$ to a conjugacy class $\mathcal{C} \subset G$ may not vanish. In this case it is impossible to describe the basic gerbe in terms of a G-invariant cover and G-equivariant transition line bundles. Compare with the case of G-equivariant line bundles over G-manifolds M: Such a line bundle may be described in terms of a G-invariant cover and G-invariant transition functions only if its pull-back to any G-orbit is equivariantly trivial.

One way of getting around this problem is to extend the Chatterjee-Hitchin theory to the equivariant case, as in [9, Appendix A]. A lift of the group action to a given gerbe is obtained by specifying the isomorphisms between the gerbe and its pull-back under the action of group elements $g \in G$. Unfortunately, the conditions for such isomorphisms to define a group action become rather complicated. A second possibility, adopted in this paper, is to use Murray's theory of *bundle gerbes* [24].

To explain our approach in more detail, let us first discuss the simplest case of $G = \mathrm{SU}(d+1)$, where it is equivalent to the construction in Gawędzki-Reis. The eigenvalues of any matrix $A \in \mathrm{SU}(d+1)$ can be uniquely written in the form

$$\exp(2\pi i \lambda_1(A)), \ldots, \exp(2\pi i \lambda_{d+1}(A))$$

where $\lambda_1(A), \ldots, \lambda_{d+1}(A) \in \mathbf{R}$ satisfy $\sum_{i=1}^{d+1} \lambda_i(A) = 0$ and

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_{d+1}(A) \geq \lambda_1(A) - 1 .$$

Define an open cover $V_1, \ldots, V_d, V_{d+1}$ of G, where V_j consists of those matrices A for which the jth inequality becomes strict. Over the set G_{reg} of regular elements, where all inequalities are strict, we have $d+1$ line bundles $L_1, \ldots, L_d, L_{d+1}$ defined by the eigenlines for the eigenvalues $\exp(2\pi i \lambda_j(A))$. For $i < j$, the tensor product $L_{i+1} \otimes \cdots \otimes L_j \to G_{\mathrm{reg}}$ extends to a line bundle $L_{ij} \to V_i \cap V_j$. (One may view L_{ij} as the top exterior power of the sum of eigenspaces for the eigenvalues in the given range.) For $i < j < k$ we have a canonical isomorphism $L_{ij} \otimes L_{jk} \cong L_{ik}$ over the triple intersection $V_i \cap V_j \cap V_k$.

The L_{ij}, together with these isomorphisms, define a gerbe over $\mathrm{SU}(d+1)$, representing the generator of $H^3(\mathrm{SU}(d+1),\mathbf{Z})$.

More generally, consider any compact, simply connected, simple Lie group G of rank d. Up to conjugacy, G contains exactly $d+1$ elements with semi-simple centralizer. (For $G=\mathrm{SU}(d+1)$, these are the central elements.) Let $\mathcal{C}_1,\ldots,\mathcal{C}_{d+1}\subset G$ be their conjugacy classes. We will define an invariant open cover $V_1,\ldots,V_{d+1}$ of G, with the property that each member of this cover admits an equivariant retraction onto the conjugacy class $\mathcal{C}_j\subset V_j$. It turns out that every semi-simple centralizer has a distinguished central extension by $\mathrm{U}(1)$. This central extension defines an equivariant bundle gerbe on $\mathcal{C}_j$, hence (by pull-back) an equivariant bundle gerbe over V_j. We will find that these gerbes over V_j glue together to produce a gerbe over G, using a gluing rule developed in this paper.

The organization of the paper is as follows. In Section 2 we review the theory of gerbes and pseudo-line bundles with connections, and discuss 'strong equivariance' under a group action. Section 4 describes gluing rules for bundle gerbes. Section 3 summarizes some facts about gerbes coming from central extensions. In Section 5 we give the construction of the basic gerbe over G outlined above, and in Section 6 we study the 'pre-quantization of conjugacy classes'.

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2. GERBES WITH CONNECTIONS

In this section we review gerbes on manifolds, along the lines of Chatterjee-Hitchin and Murray.

2.1 CHATTERJEE-HITCHIN GERBES

Let M be a manifold. Any Hermitian line bundle over M can be described by an open cover U_a, and transition functions $\chi_{ab}\colon U_a\cap U_b\to \mathrm{U}(1)$ satisfying a cocycle condition $(\delta\chi)_{abc}=\chi_{bc}\chi_{ac}^{-1}\chi_{ab}=1$ on triple intersections. The cohomology class in $H^1(M,\underline{\mathrm{U}(1)}) = H^2(M,\mathbf{Z})$ defined by this cocycle is the Chern class of the line bundle. Chatterjee-Hitchin [10, 18, 17] suggested to realize classes in $H^3(M,\mathbf{Z})$ in a similar fashion, replacing U(1)-valued functions with Hermitian line bundles. They define a gerbe to be a collection of Hermitian transition line bundles $L_{ab} \to U_a \cap U_b$ and a trivialization, i.e. unit length section, t_{abc} of the line bundle $(\delta L)_{abc} = L_{bc}L_{ac}^{-1}L_{ab}$ over triple intersections. These trivializations have to satisfy a compatibility relation over quadruple intersections,

$$(\delta t)_{abcd} \equiv t_{bcd}t_{acd}^{-1}t_{abd}t_{abc}^{-1} = 1\,,$$

which makes sense since $(\delta t)_{abcd}$ is a section of the *canonically* trivial bundle. (Each factor L_{ab} cancels with a factor L_{ab}^{-1}.) After passing to a refinement of the cover, such that all L_{ab} become trivializable, and picking trivializations, t_{abc} is simply a Čech cocycle of degree 2, hence defines a class in $H^2(M,\underline{\mathrm{U}(1)}) = H^3(M,\mathbf{Z})$. The class is independent of the choices made in this construction, and is called the *Dixmier-Douady class* of the gerbe.

Note that in practice, it is often not desirable to pass to a refinement. For example, if M is a connected, oriented 3-manifold, the generator of $H^3(M,\mathbf{Z}) = \mathbf{Z}$ can be described in terms of the cover U_1, U_2, where U_1 is an open ball around a given point $p \in M$, and $U_2 = M\backslash\{p\}$, using the degree one line bundle over $U_1 \cap U_2 \cong S^2 \times (0,1)$.

2.2 Bundle gerbes

Bundle gerbes were invented by Murray [24], generalizing the following construction of line bundles. Let $\pi\colon X \to M$ be a fiber bundle, or more generally a surjective submersion. (Different components of X may have different dimensions.) For each $k \geq 0$ let $X^{[k]}$ denote the k-fold fiber product of X with itself. There are $k+1$ projections $\partial^i\colon X^{[k+1]} \to X^{[k]}$, omitting the ith factor in the fiber product. Suppose we are given a smooth function $\chi\colon X^{[2]} \to \mathrm{U}(1)$, satisfying a cocycle condition $\delta\chi = 1$ where

$$\delta\chi := \partial_0^*\chi\partial_1^*\chi^{-1}\partial_2^*\chi\colon X^{[3]} \to \mathrm{U}(1)\,.$$

Then χ determines a Hermitian line bundle $L \to M$, with fibers at $m \in M$ the space of all linear maps $\phi\colon X_m = \pi^{-1}(m) \to \mathbf{C}$ such that $\phi(x) = \chi(x,x')\phi(x')$. Given local sections $\sigma_a\colon U_a \to X$ of X, the pull-backs of χ under the maps $(\sigma_a,\sigma_b)\colon U_a \cap U_b \to X^{[2]}$ give transition functions χ_{ab} for the line bundle.

Again, replacing U(1)-valued functions by line bundles in this construction, one obtains a model for gerbes: A bundle gerbe is given by a line bundle $L \to X^{[2]}$ and a trivializing section t of the line bundle $\delta L = \partial_0^*L\otimes\partial_1^*L^{-1}\otimes\partial_2^*L$

over $X^{[3]}$, satisfying a compatibility condition $\delta t = 1$ over $X^{[4]}$ (which makes sense since δt is a section of the canonically trivial bundle $\delta\delta L$). Given local sections $\sigma_a\colon U_a \to X$, one can pull these data back under the maps $(\sigma_a, \sigma_b)\colon U_a \cap U_b \to X^{[2]}$ and $(\sigma_a, \sigma_b, \sigma_c)\colon U_a \cap U_b \cap U_c \to X^{[3]}$ to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of (X, L, t) is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with X the disjoint union of the sets U_a in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called 'locally split') is used that every point $x \in M$ admits an open neighborhood U and a map $\sigma\colon U \to X$ such that $\pi \circ \sigma = \mathrm{id}$. However, this condition seems insufficient in the smooth category, as the fiber product $X \times_M X$ need not be a manifold unless π is a submersion.

2.3 SIMPLICIAL GERBES

Murray's construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik's notes of lectures by R. Bott [23] and to Dupont's paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold* $M_\bullet$ is a sequence of manifolds $(M_n)_{n=0}^{\infty}$, together with *face maps* $\partial_i\colon M_n \to M_{n-1}$ for $i = 0, \ldots, n$ satisfying relations $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ for $i < j$. (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of $M_\bullet$ is the topological space $||M|| = \coprod_{n=1}^{\infty} \Delta^n \times M_n / \sim$, where Δ^n is the n-simplex and the relation is $(t, \partial_i(x)) \sim (\partial^i(t), x)$, for $\partial^i\colon \Delta^{n-1} \to \Delta^n$ the inclusion as the ith face. A (smooth) simplicial map between simplicial manifolds $M_\bullet, M'_\bullet$ is a collection of smooth maps $f_n\colon M_n \to M'_n$ intertwining the face maps; such a map induces a map between the geometric realizations.

EXAMPLES 2.2.

(a) If S is any manifold, one can define a simplicial manifold $E_\bullet S$ where $E_n S$ is the $n+1$-fold cartesian product of S, and ∂_j omits the jth factor. It is known [23] that the geometric realization $||ES||$ of this simplicial manifold is contractible. More generally, if $X \to M$ is a fiber bundle with fiber S, one can define a simplicial manifold $E_nX := X^{[n+1]}$, with face maps as in Section 2.2. The geometric realization $||EX||$ becomes a fiber bundle over M with contractible fiber $||ES||$.

(b) [22, 27] For any Lie group G there is a simplicial manifold $B_nG = G^n$. The face maps ∂_i for $0 < i < n$ are

$$\partial_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

while ∂_0 omits the first component and ∂_n the last component. The map $\pi_n\colon E_nG \to B_nG$ given by $\pi_n(k_0, \ldots, k_n) = (k_0k_1^{-1}, \ldots, k_{n-1}k_n^{-1})$ is simplicial, and the induced map on geometric realizations is a model for the classifying bundle $EG \to BG$.

(c) [27, 23] If $\mathcal{U} = \{U_a, a \in A\}$ is an open cover of M, one defines a simplicial manifold

$$\mathcal{U}_nM := \coprod_{(a_0,\ldots,a_n)\in A_n} U_{a_0\ldots a_n}$$

where A_n is the set of all sequences $(a_0, \ldots, a_n)$ such that $U_{a_0\ldots a_n} := U_{a_0} \cap \ldots \cap U_{a_n}$ is non-empty. The face maps are induced by the inclusions,

$$\partial_i\colon U_{a_0\ldots a_n} \hookrightarrow U_{a_0\ldots\widehat{a_i}\ldots a_n}.$$

One may view this as a special case of (a), with $X = \coprod_{a\in A} U_a$. It is known [23, Theorem 7.3] that $||\mathcal{U}M||$ is homotopy equivalent to M.

(d) [2] The definitions of E_nG and B_nG extend to Lie groupoids G over a base S. If $s, t\colon G \to S$ are the source and target maps, one defines E_nG as the $n+1$-fold fiber product of G with respect to the target map t. The space B_nG for $n \geq 1$ is the set of all $(g_1, \ldots, g_n) \in G^n$ with $s(g_j) = t(g_{j-1})$, while $B_0G = S$. The definition of the face maps $\partial_j\colon B_nG \to B_{n-1}G$ is as before for $n > 1$, while for $n = 1$, $\partial_0 = t$ and $\partial_1 = s$. We have a simplicial map $E_nG \to B_nG$ defined just as in the group case.

The bi-graded space of differential forms $\Omega^\bullet(M_\bullet)$ carries two commuting differentials d, δ, where d is the de Rham differential and $\delta\colon \Omega^k(M_n) \to \Omega^k(M_{n+1})$ is an alternating sum, $\delta\alpha = \sum_{i=0}^{n+1}(-1)^i\partial_i^*\alpha$. It is known [23, Theorem 4.2, Theorem 4.5] that the total cohomology of this double complex is the (singular) cohomology of the geometric realization, with coefficients in $\mathbf{R}$.

We will use the δ notation in many similar situations: For instance, given a Hermitian line bundle $L \to M_n$, we define a Hermitian line bundle $\delta L \to M_{n+1}$ as a tensor product,

$$\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \cdots \otimes \partial_{n+1}^* L^{\pm} .$$

The line bundle $\delta(\delta L) \to M_{n+1}$ is canonically trivial, due to the relations between face maps. If σ is a unitary section (i.e. a trivialization) of L, one uses a similar formula to define a unitary section $\delta\sigma$ of δL. Then $\delta(\delta\sigma) = 1$ (the identity section of the trivial line bundle $\delta(\delta L)$). For any unitary connection ∇ of L, one defines a unitary connection $\delta\nabla$ of δL in the obvious way.

CONVENTION. For the rest of this paper, we take all line bundles L to be *Hermitian* line bundles, and all connections ∇ on L to be *unitary* connections.

Let $M_\bullet$ be a simplicial manifold. One might define a simplicial line bundle as a collection of line bundles $L_n \to M_n$ such that the face maps $\partial_i \colon M_n \to M_{n-1}$ lift to line bundle homomorphisms $\hat{\partial}_i \colon L_n \to L_{n-1}$, satisfying the face map relations. Thus $L_\bullet$ is itself a simplicial manifold, and its geometric realization $||L||$ is a line bundle over $||M||$. Equivalently, the lifts $\hat{\partial}_i$ may be viewed as isomorphisms, $\partial_i^* L_{n-1} \to L_n$. In particular, we may identify L_n with the pull-back of $L := L_0$ under the nth-fold iterate $\partial_0 \circ \cdots \circ \partial_0$.

The isomorphisms $\partial_1^* L \cong \partial_0^* L = L_1$ determine a unitary section t of $\delta L \to M_1$, and the compatibility of isomorphisms

$$(\partial_0\partial_2)^* L \cong (\partial_0\partial_1)^* L \cong (\partial_0\partial_0)^* L = L_2$$

amount to the condition $\delta t = 1$. (Compatibility of the isomorphisms for L_n with $n \geq 3$ is then automatic.) That is, *a simplicial line bundle over $M_\bullet$ is given by a line bundle $L \to M_0$, together with a unitary section t of $\delta L \to M_1$, such that $\delta t = 1$ over M_2.* A unitary section s of L with $\delta s = t$ induces a unitary section of $||L|| \to ||M||$.

Taking L to be trivial, we see in particular that any U(1)-valued function t on M_1, with $\delta t = 1$, defines a line bundle over the geometric realization. A trivialization of that line bundle is given by a U(1)-valued function on M_0 satisfying $\delta s = t$. Replacing U(1)-valued functions with line bundles, this motivates the following definition.

DEFINITION 2.3. A *simplicial gerbe* over $M_\bullet$ is a pair (L, t), consisting of a line bundle $L \to M_1$, together with a section t of $\delta L \to M_2$ satisfying $\delta t = 1$. A pseudo-line bundle for (L, t) is a pair (E, s), consisting of a line bundle $E \to M_0$ and a section s of $\delta E^{-1} \otimes L$ such that $\delta s = t$.

REMARK 2.4.

(a) We are using the notion of a simplicial gerbe only as a 'working definition'. It is clear from the discussion above that a more general notion would involve a gerbe over M_0.

(b) In [9], what we call simplicial gerbe is called a simplicial line bundle. The name pseudo-line bundle is adopted from [9], where it is used in a similar context.

A simplicial gerbe over $\mathcal{U}_\bullet M$ (for a cover $\mathcal{U}$ of M) is a Chatterjee-Hitchin gerbe, while a simplicial gerbe over $E_\bullet X = X^{[\bullet+1]}$ (for a surjective submersion $X \to M$) is a bundle gerbe. It is shown in [24] that the characteristic class of a bundle gerbe (X, L, t) vanishes if and only if it admits a pseudo-line bundle.

EXAMPLE 2.5 (Central extensions). (See [9, p. 615].) Let K be a Lie group. A simplicial line bundle over $B_\bullet K$ is the same thing as a group homomorphism $K \to \mathrm{U}(1)$: The line bundle $L \to B_0 K$ is trivial since $B_0 K$ is just a point, hence the unitary section t of δL becomes a $\mathrm{U}(1)$-valued function. The condition $\delta t = 1$ means that this function is a group homomorphism.

Similarly, a simplicial gerbe (Γ, τ) over $B_\bullet K$ is the same thing as a central extension

$$\mathrm{U}(1) \to \widehat{K} \to K.$$

Indeed, given the line bundle $\Gamma \to K$ let $\widehat{K}$ be the unit circle bundle inside Γ. The fiber of $\delta\Gamma \to K^2$ at (k_1, k_2) is a tensor product $\Gamma_{k_2}\Gamma_{k_1k_2}^{-1}\Gamma_{k_1}$, hence the section τ of $\delta\Gamma \to K^2$ defines a unitary isomorphism $\Gamma_{k_1}\Gamma_{k_2} \cong \Gamma_{k_1k_2}$, or equivalently a product on $\widehat{K}$ covering the group multiplication on K. Finally, the condition $\delta\tau = 1$ is equivalent to associativity of this product.

A pseudo-line bundle (E, s) for the simplicial gerbe (Γ, τ) is the same thing as a splitting of the central extension: Obviously E is trivial since $B_0 K$ is just a point; the section s defines a trivialization $\widehat{K} = K \times \mathrm{U}(1)$, and $\delta s = t$ means that this is a group homomorphism.

DEFINITION 2.6. A connection on a simplicial gerbe (L, t) over $M_\bullet$ is a line bundle connection ∇^L, together with a 2-form $B \in \Omega^2(M_0)$, such that $(\delta\nabla^L)\, t = 0$ and

$$\delta B = \frac{1}{2\pi i}\,\mathrm{curv}(\nabla^L).$$

Given a pseudo-line bundle $\mathcal{L} = (E, s)$, we say that ∇^E is a pseudo-line bundle connection if it has the property $((\delta\nabla^E)^{-1}\nabla^L)s = 0$.

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the δ-cohomology of the double complex $\Omega^k(M_n)$ vanishes in bidegrees $(1,2)$ and $(2,1)$. In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion $\pi\colon X \to M$ the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \cdots \tag{2.1}$$

is exact, so the δ-cohomology vanishes in *all* degrees.

Thus, every bundle gerbe $\mathcal{G} = (X, L, t)$ over a manifold M (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the 3*-curvature* $\eta \in \Omega^3(M)$ of the bundle gerbe connection by $\pi^*\eta = \mathrm{d}B \in \ker\delta$. It can be shown that its cohomology class is the image of the Dixmier-Douady class $[\mathcal{G}]$ under the map $H^3(M,\mathbf{Z}) \to H^3(M,\mathbf{R})$. Similarly, if $\mathcal{G}$ admits a pseudo-line bundle $\mathcal{L} = (E, s)$, one can always choose a pseudo-line bundle connection ∇^E. The difference $\frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B$ is δ-closed and one defines the *error* 2*-form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i}\operatorname{curv}(\nabla^E) - B\,.$$

It is clear from the definition that $\mathrm{d}\omega + \eta = 0$.

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

2.4 Equivariant bundle gerbes

Suppose G is a Lie group acting on X and on M, and that $\pi\colon X \to M$ is a G-equivariant surjective submersion. Then G acts on all fiber products $X^{[p]}$. We will say that a bundle gerbe $\mathcal{G} = (X, L, t)$ is G*-equivariant*, if L is a G-equivariant line bundle and t is a G-invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction[1]) $X_G = EG \times_G X \to M_G = EG \times_G M$, hence has an *equivariant* Dixmier-Douady class in $H^3(M_G,\mathbf{Z}) = H^3_G(M,\mathbf{Z})$. Similarly, we say that a pseudo-line bundle (E,s) for (X,L,t) is equivariant, provided E carries a G-action and s is an invariant section.

[1]) We have not discussed bundle gerbes over infinite-dimensional spaces such as M_G. Recall however [4] that the classifying bundle $EG \to BG$ may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G-action on X would amount to the cover being G-invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G, the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G-equivariant polynomial maps $\alpha\colon \mathfrak{g} \to \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(\mathrm{d}_G\, \alpha)(\xi) = \mathrm{d}\, \alpha(\xi) - \iota(\xi_M)\alpha(\xi)\,,$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G-equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G-closed equivariant curvature $\mathrm{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G-equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G), where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta\nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^*\eta_G = \mathrm{d}_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s), one defines the equivariant error 2-form ω_G by

$$\pi^*\omega_G = \frac{1}{2\pi i}\mathrm{curv}_G(\nabla^E) - B_G\,.$$

Clearly, $\mathrm{d}_G\,\omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G. Suppose $\mathrm{U}(1) \to \widehat{K} \to K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K-bundle $\pi\colon P \to B$, one constructs a bundle gerbe (P, L, t), sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K\,,$$

which we may view as a fiber bundle over B but also as a fiber bundle $E_nK \times_K P$ over B_nK. Let

$$f_\bullet \colon E_\bullet P \to B_\bullet K \tag{3.1}$$

be the bundle projection. Then $L = f_1^*\Gamma$, $t = f_2^*\tau$ defines a bundle gerbe (P, L, t). A pseudo-line bundle for this bundle gerbe is equivalent to a lift of the structure group to $\widehat{K}$: Indeed if $\widehat{P}$ is a principal $\widehat{K}$-bundle lifting P, consider the associated bundle $E = \widehat{P} \times_{\mathrm{U}(1)} \mathbf{C}$. From the action map $\widehat{K} \times \widehat{P} \to \widehat{P}$ one obtains an isomorphism $\Gamma_k \otimes E_p \cong E_{k.p}$, or equivalently a section s of $\delta E^{-1} \otimes L$. One checks that $\delta s = t$, so that (E, s) is a pseudo-line bundle. Conversely, the bundle $\widehat{P}$ is recovered as the unit circle bundle in E, and s defines an action of $\widehat{K}$ lifting the action of K. See Gomi [14] for a detailed construction of bundle gerbe connections on (P, L, t).

REMARK 3.1. To obtain a Chatterjee-Hitchin gerbe from this bundle gerbe, we must choose a cover $\mathcal{U}$ of M such that P is trivial over each $U_a \in \mathcal{U}$. Any choice of trivialization gives a simplicial map $\mathcal{U}_\bullet M \to E_\bullet P$, and we pull back the bundle gerbe under this map. More directly, the local trivializations give rise to a 'classifying map' $\chi_\bullet \colon \mathcal{U}_\bullet M \to B_\bullet K$ (see [23]), and the Chatterjee-Hitchin gerbe is defined as the pull-back of (Γ, τ) under this map.

Suppose the group K is compact and connected. After pulling back to the universal cover $\widetilde{K}$, every central extension $\mathrm{U}(1) \to \widehat{K} \to K$ becomes trivial. It follows that every central extension of K by $\mathrm{U}(1)$ is of the form

$$\widehat{K} = \widetilde{K} \times_{\pi_1(K)} \mathrm{U}(1),$$

where $\pi_1(K) \subset \widetilde{K}$ acts on $\mathrm{U}(1)$ via some homomorphism $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$. The choice of ϱ for a given extension is equivalent to the choice of a flat $\widehat{K}$-invariant connection on the principal $\mathrm{U}(1)$-bundle $\widehat{K} \to K$. The central extension is isomorphic to the *trivial* extension if and only if ϱ extends to a homomorphism $\widetilde{\varrho} \colon \widetilde{K} \to \mathrm{U}(1)$, and the choice of any such $\widetilde{\varrho}$ is equivalent to a choice of trivialization. Using the natural map from $(\mathfrak{k}^*)^K = \mathrm{Hom}(\widetilde{K}, \mathbf{R})$ onto $\mathrm{Hom}(\widetilde{K}, \mathrm{U}(1))$ this gives an exact sequence of Abelian groups

$$(\mathfrak{k}^*)^K \to \mathrm{Hom}(\pi_1(K), \mathrm{U}(1)) \to \{\text{central extensions of } K \text{ by } \mathrm{U}(1)\} \to 1\,. \tag{3.2}$$

Suppose K is semi-simple (so that $(\mathfrak{k}^*)^K = 0$), and T is a maximal torus in K. Let $\widetilde{T} \subset \widetilde{K}$ be the maximal torus given as the pre-image of T. Let $\Lambda_K, \widetilde{\Lambda}_K \subset \mathfrak{t}$ be the integral lattices of $T, \widetilde{T}$. The lattice $\widetilde{\Lambda}_K$ is equal to the co-root lattice of K, and $\pi_1(K) = \widetilde{\Lambda_K}/\Lambda_K$ (cf. [6, Theorem V.7.1]). Therefore, if K is semi-simple,

$$\{\text{central extensions of } K \text{ by } \mathrm{U}(1)\} = \mathrm{Hom}(\pi_1(K), \mathrm{U}(1)) = \widetilde{\Lambda}_K^*/\Lambda_K^*,$$

the quotient of the dual of the co-root lattice by the weight lattice.

PROPOSITION 3.2. *Suppose K is a compact, connected Lie group and $\pi\colon P \to M$ a principal K-bundle.*

(a) *Any $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$ defines a bundle gerbe (P, L, t) over M, together with a gerbe connection (∇^L, B) where $B = 0$. In particular this gerbe is* flat.

(b) *If ϱ is the image of $\mu \in (\mathfrak{k}^*)^K$, there is a distinguished pseudo-line bundle $\mathcal{L} = (E, s)$ for this gerbe, with E a trivial line bundle. Any principal connection $\theta \in \Omega^1(P, \mathfrak{k})$ defines a connection on $\mathcal{L}$, with error 2-form $\omega \in \Omega^2(M)$ given by $\pi^*\omega = \langle \mu, F^\theta \rangle \in \Omega^2(M)$, where F^θ is the curvature.*

Proof. Let $\mathrm{U}(1) \to \widehat{K} \to K$ be the central extension defined by ϱ, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. As remarked above, ϱ defines a flat connection on $\widehat{K} \to K$, hence also a flat connection ∇^Γ on the line bundle $\Gamma \to B_1 K$. Then $(\nabla^\Gamma, 0)$ is a connection on the simplicial gerbe (Γ, τ). Pulling back under the map $f_\bullet$ (cf. (3.1)) we obtain a connection $(\nabla^L, 0)$ on the bundle gerbe (P, L, t).

If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, the corresponding trivialization of $\widehat{K}$ defines a unitary section σ of Γ, with $\delta\sigma = \tau$ and $\frac{1}{2\pi i}\nabla^\Gamma \sigma = \langle \mu, \theta^L \rangle \sigma$, where θ^L is the left-invariant Maurer-Cartan form on K. Thus $\mathcal{L} = (E, s)$, with E the trivial line bundle and $s = f_1^*\sigma$, is a pseudo-line bundle for $\mathcal{G}$. Given a principal connection θ, let ∇^E be the connection on the trivial bundle E, having connection 1-form $\langle \mu, \theta \rangle \in \Omega^1(P)$. Since $\frac{1}{2\pi i}\nabla^L s = f_1^* \langle \mu, \theta^L \rangle s$, it follows that

$$\frac{1}{2\pi i}((\delta\nabla^E)^{-1}\nabla^L)s = \langle \mu, f_1^*\theta^L - \delta\theta \rangle . \tag{3.3}$$

One finds $\partial_1^*\theta = \mathrm{Ad}_{f_1^{-1}}(\partial_0^*\theta - f_1^*\theta^L)$. Since μ is K-invariant, this shows that the right hand side of (3.3) vanishes. Thus ∇^E is a pseudo-line bundle connection. The error 2-form ω is given by

$$\pi^*\omega = \mathrm{d}\,\langle \mu, \theta \rangle = \langle \mu, \mathrm{d}\,\theta \rangle = \langle \mu, F^\theta \rangle .$$

All of these constructions can be made equivariant in a rather obvious way: Thus if G is another Lie group and P is a G-invariant principal K-bundle, any $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$ defines a G-equivariant bundle gerbe (P, L, t) (with flat connection) over M. If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, there is a G-equivariant pseudo-line bundle for this gerbe. Furthermore any choice of G-equivariant principal connection on P defines a G-equivariant pseudo-line bundle connection, with equivariant error 2-form $\pi^*\omega_G = \langle \mu, F_G^\theta \rangle$ where $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$ is the equivariant curvature.

4. Gluing data

In this Section we describe a procedure for gluing a collection of bundle gerbes (X_i, L_i, t_i) on open subsets $V_i \subset M$, with pseudo-line bundles of their quotients on overlaps[2]). We begin with the somewhat simpler case that the surjective submersions $X_i \to V_i$ are obtained by restricting a surjective submersion $X \to M$, and later reduce the general case to this special case.

Thus, let $\pi \colon X \to M$ be a surjective submersion and let V_i, $i = 0, \dots, d$ an open cover of M. Let $X_i = X|_{V_i}$, and more generally $X_I = X|_{V_I}$ where V_I is the intersection of all V_i with $i \in I$.

Suppose we are given bundle gerbes (X_i, L_i, t_i) over V_i and pseudo-line bundles (E_{ij}, s_{ij}) for the quotients $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$ over $V_i \cap V_j$, where $E_{ij} = E_{ji}^{-1}$ and $s_{ij} = s_{ji}^{-1}$. Note that $E_{ij} E_{jk} E_{ki}$ is a pseudo-line bundle for the trivial gerbe, hence is a pull-back $\pi^* F_{ijk}$ of a line bundle $F_{ijk} \to M$, and we will also require a unitary section u_{ijk} of that line bundle. Under suitable conditions the data (E_{ij}, s_{ij}) and u_{ijk} can be used to 'glue' the gerbes (X_i, L_i, t_i). The glued gerbe will be defined over the disjoint union $\coprod_{i=1}^d X_i$. We have

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[2]} = \coprod_{ij} X_i \times_M X_j$$

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[3]} = \coprod_{ijk} X_i \times_M X_j \times_M X_k$$

$$\dots$$

Hence, the glued gerbe will be of the form $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ where L_{ij} are line bundles over $X_i \times_M X_j$ and t_{ijk} unitary sections of a line bundle $(\delta L)_{ijk}$

2) See Stevenson [29] for similar gluing constructions.

over $\coprod_{ijk} X_i \times_M X_j \times_M X_k$. We will define L_{ij} by tensoring $L_i \to X^{[2]}$ (restricted to $X_i \times_M X_j$) with the pull-back of E_{ij} under the map $\partial_1 \colon X_i \times_M X_j \to X_{ij}$.

PROPOSITION 4.1. *Suppose the sections u_{ijk} satisfy the cocycle condition $u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1} = 1$, and the sections s_{ij} satisfy a cocycle condition $s_{ij}s_{jk}s_{ki} = 1$. Then there is a well-defined gerbe $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ over M, where $L_{ij} \to X_i \times_M X_j$ is the line bundle*

$$L_{ij} = L_j \otimes \partial_1^* E_{ij}$$

and t_{ijk} is a section of $(\delta L)_{ijk} \to X_i \times_M X_j \times_M X_k$ given by

$$t_{ijk} = t_k \otimes \partial_2^* s_{kj} \otimes \partial_2^* \partial_1^* \pi^* u_{ijk}\,. \tag{4.1}$$

Proof. A short calculation gives

$$(\delta L)_{ijk} = (\delta L_k) \otimes \partial_2^*(L_j L_k^{-1} \delta E_{kj}^{-1}) \otimes \partial_2^* \partial_1^* \pi^* F_{ijk}\,,$$

showing that t_{ijk} is a well-defined section of $(\delta L)_{ijk}$. One finds furthermore

$$\begin{aligned}(\delta t)_{ijkl} &= (\delta t_l) \otimes \partial_3^*\left(t_l t_k^{-1} \delta s_{kl}^{-1} \otimes \partial_2^*\left(s_{lj}s_{jk}s_{kl} \otimes \partial_1^* \pi^*(u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1})\right)\right)\\ &= \partial_3^*\partial_2^*\left(s_{lj}s_{jk}s_{kl} \otimes \partial_1^*\pi^*(u_{jkl}u_{ikl}^{-1}u_{ijl}u_{ijk}^{-1})\right)\end{aligned}$$

which equals 1 under the given assumptions on u and s.

The gluing construction described in this Proposition is particularly natural for Chatterjee-Hitchin gerbes: Suppose $\mathcal{U}$ is an open cover of M, and $X = \coprod_{U \in \mathcal{U}} U$. For any decomposition $\mathcal{U} = \coprod_{i=1}^d \mathcal{U}_i$ let $V_i = \cup_{U \in \mathcal{U}_i} U$, and $X_i = \coprod_{U \in \mathcal{U}_i} U$. Note that in this case,

$$\coprod_i X_i = X\,.$$

Suppose (L_i, t_i) are Chatterjee-Hitchin gerbes for the cover $\mathcal{U}_i$ of V_i, and that we are given pseudo-line bundles (E_{ij}, s_{ij}) and a section u_{ijk} as above. Note that the E_{ij} are a collection of line bundles over intersections $U_a \cap U_b$ where $U_a \in \mathcal{U}_i$ and $U_b \in \mathcal{U}_j$. The gluing construction gives a Chatterjee-Hitchin gerbe (L, t) for the cover $\mathcal{U}$ of M, where the E_{ij} enter the definition of transition line bundles between open sets in distinct $\mathcal{U}_i$, $\mathcal{U}_j$.

REMARK 4.2. Suppose $X = M$, and that all L_i, t_i, s_{ij} are trivial. Then the gerbe described in Proposition 4.1 *is* a Chatterjee-Hitchin gerbe for the cover $\{V_i\}$. The E_{ij} now play the role of transition line bundles, and u_{ijk} play the role of t.

Suppose now that, in addition to the assumptions of Proposition 4.1, we have gerbe connections (∇^{L_i}, B_i) and pseudo-line bundle connections $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$. Let ω_{ij} denote the error 2-form for $\nabla^{E_{ij}}$.

PROPOSITION 4.3. *The connections $\nabla^{L_{ij}} = \nabla^{L_j} \otimes \partial_1^* \nabla^{E_{ij}}$ on L_{ij}, together with the two forms $B_i \in \Omega^2(X_i)$, define a gerbe connection if all error 2-forms ω_{ij} vanish, and if*

$$\nabla^{E_{ij}} \nabla^{E_{jk}} \nabla^{E_{ki}} (\pi^* u_{ijk}) = 0 .$$

Proof. Let B be the 2-form on $\coprod X_i$ given by B_i on X_i. We first verify that $\frac{1}{2\pi i} \operatorname{curv}(\nabla^{L_{ij}}) = (\delta B)_{ij}$:

$$\begin{aligned}
\frac{1}{2\pi i} \operatorname{curv}(\nabla^{L_{ij}}) &= \frac{1}{2\pi i} \operatorname{curv}(\nabla^{L_j}) + \frac{1}{2\pi i} \partial_1^* \operatorname{curv}(\nabla^{E_{ij}}) \\
&= \delta B_j + \partial_1^* (B^j - B^i + \pi^* \omega_{ij}) \\
&= \partial_0^* B_j - \partial_1^* B_i = (\delta B)_{ij} .
\end{aligned}$$

Next, we check that t_{ijk} is parallel for $(\delta \nabla^L)_{ijk}$:

$$\begin{aligned}
(\delta \nabla^L)_{ijk} &= \partial_0^* \nabla^{L_{jk}} \partial_1^* (\nabla^{L_{ik}})^{-1} \partial_2^* \nabla^{L_{ij}} \\
&= \delta \nabla^{L_k} \otimes \partial_2^* (\nabla^{L_k} (\nabla^{L_j})^{-1} \delta \nabla^{E_{jk}}) \otimes \partial_2^* \partial_1^* (\nabla^{E_{ij}} \nabla^{E_{jk}} \nabla^{E_{ki}}) .
\end{aligned}$$

This annihilates (4.1) as required.

We now describe a slightly more complicated gluing construction, in which the X_i are not simply the restrictions of a surjective submersion $X \to M$. Instead, we assume that for each I we are given a surjective submersion $\pi_I \colon X_I \to V_I$ are surjective submersions, and for each $I \supset J$ a fiber preserving smooth map $f_I^J \colon X_I \to X_J$, with the compatibility condition $f_J^K \circ f_I^J = f_I^K$ for $I \supset J \supset K$. Our gluing data will consist of the following:

(i) Over each V_i, bundle gerbes (X_i, L_i, t_i) with connections (∇^{L_i}, B_i).

(ii) Over each V_{ij}, pseudo-line bundles $E_{ij} = E_{ji}^{-1}, s_{ij} = s_{ji}^{-1}$ with connections $\nabla^{E_{ij}} = (\nabla^{E_{ji}})^{-1}$ for the bundle gerbe (X_{ij}, L_{ij}, t_{ij}), given as the quotient of the pull-back of (X_j, L_j, t_j) by f_{ij}^j and the pull-back of (X_i, L_i, t_i) by f_{ij}^i.

(iii) Over triple intersections, unitary sections u_{ijk} of the line bundle $F_{ijk} \to V_{ijk}$ defined by tensoring the pull-backs of E_{ij}, E_{jk}, E_{ki} by the maps f_{ijk}^{ij}, f_{ijk}^{jk}, f_{ijk}^{ki}.

We require that the s_{ij} and u_{ijk} satisfy a cocycle condition similar to Proposition 4.1, that all error 2-forms ω_{ij} are zero, and that the connections $\nabla^{E_{ij}}$ satisfy a compatibility condition as in 4.3.

These data may be used to define a bundle gerbe over M, by reducing to the setting of Propositions 4.1, 4.3. As a first step we construct a more convenient cover.

LEMMA 4.4. *There are open subsets U_I of M, with $\overline{U}_I \subset V_I$, and $\bigcup_I U_I = M$, such that*

$$\overline{U}_I \cap \overline{U}_J = \emptyset \quad \textit{unless} \quad J \subset I \textit{ or } I \subset J\,.$$

The collection of open subsets

$$V_i' = M \backslash \bigcup_{J \not\ni i} \overline{U}_J$$

is a shrinking of the open cover V_i, that is, $\bigcup V_i' = M$ and $\overline{V_i'} \subset V_i$.

The proof of this technical lemma is deferred to Appendix A. Now set $X = \coprod_I X_I|_{U_I}$. By definition of V_i', the restriction $X_i' = X|_{V_i'}$ is given by

$$X_i' = \coprod_{J \ni i} X_J|_{U_J \cap V_i'}\,.$$

More generally, letting $V_I' = \bigcap_{i \in I} V_i'$ and $X_I' = X|_{V_I'}$ we have

$$X_I' = \coprod_{J \supset I} X_J|_{U_J \cap V_I'}\,.$$

Let $X_I' \to X_I|_{V_I'}$ be the fiber preserving map, given on $X_J|_{U_J \cap V_I'}$ by the map $f_J^I \colon X_J \to X_I$. Using these maps, we can pull-back our gluing data: Let (X_i', L_i', t_i') be the pull-back of the bundle gerbe (X_i, L_i, t_i) under the map $X_i' \to X_i$, equipped with the pull-back connection. On overlaps V_{ij}', we let (E_{ij}', s_{ij}') be the pseudo-line bundle with connections defined by pulling back (E_{ij}, s_{ij}). The gluing data obtained in this way satisfy the conditions from Propositions 4.1 and 4.3, and hence give rise to a bundle gerbe with connection over M.

REMARK 4.5. In our applications, the line bundles E_{ij} are in fact trivial, so one can simply take $u_{ijk} = 1$ in terms of the trivialization. The s_{ij} are U(1)-valued functions in this case, and the compatibility condition reads $s_{ij}s_{jk}s_{ki} = 1$ over X_{ijk}.

The gluing constructions generalize equivariant bundle gerbes in a straightforward way.

5. The basic gerbe over a compact simple Lie group

In this section we explain our construction of the basic gerbe over a compact, simple, simply connected Lie group.

5.1 Notation

Let G be a compact, simple, simply connected Lie group, with Lie algebra $\mathfrak{g}$. For any action of $G \times M \to M$, $(g,m) \mapsto g.m$ on a manifold M, we will denote by G_m the stabilizer group of a point $m \in M$. If $M = G$ or $M = \mathfrak{g}$, we will always consider the adjoint action of G unless specified otherwise. For instance, G_g for denotes the centralizer of an element $g \in G$.

Choose a maximal torus T of G, with Lie algebra $\mathfrak{t}$. Let $\Lambda = \ker(\exp|_{\mathfrak{t}})$ be the integral lattice and $\Lambda^* \subset \mathfrak{t}^*$ its dual, the (real) weight lattice. Equivalently, Λ is characterized as the lattice generated by the coroots $\check{\alpha}$ for the (real) roots α. Recall that the *basic inner product* $\cdot$ on $\mathfrak{g}$ is the unique invariant inner product such that $\check{\alpha} \cdot \check{\alpha} = 2$ for all long roots α. Throughout this paper, we will use the basic inner product to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Choose a collection of simple roots $\alpha_1, \ldots, \alpha_d \in \Lambda^*$ and let $\mathfrak{t}_+ = \{\xi \mid \alpha_j \cdot \xi \geq 0,\ j = 1, \ldots, d\}$ be the corresponding positive Weyl chamber. The fundamental alcove $\mathfrak{A}$ is the subset cut out from $\mathfrak{t}_+$ by the additional inequality $\alpha_0 \cdot \xi \geq -1$ where α_0 is the lowest root.

The fundamental alcove parametrizes conjugacy classes in G, in the sense that each conjugacy class contains a unique point $\exp \xi$ with $\xi \in \mathfrak{A}$. The quotient map will be denoted $q \colon G \to \mathfrak{A}$. Let $\mu_0, \ldots, \mu_d$ be the vertices of $\mathfrak{A}$, with $\mu_0 = 0$. For any $I \subseteq \{0, \ldots, d\}$, all group elements $\exp \xi$ with ξ in the open face spanned by μ_j with $j \in I$ have the same centralizer, denoted G_I. In particular, G_j will denote the centralizer of $\exp \mu_j$.

For each j let $\mathfrak{A}_j \subset \mathfrak{A}$ be the open star at μ_j, i.e. the union of all open faces containing μ_j in their closure. Put differently, $\mathfrak{A}_j$ is the complement of the closed face opposite to the vertex μ_j. We will work with the open cover of G given by the pre-images, $V_j = q^{-1}(\mathfrak{A}_j)$. More generally let $\mathfrak{A}_I = \bigcap_{j \in I} \mathfrak{A}_j$, and $V_I := q^{-1}(\mathfrak{A}_I)$. The flow-out $S_I = G_I . \exp(\mathfrak{A}_I)$ of $\exp(\mathfrak{A}_I) \subset T$ under the action of G_I is an open subset of G_I, and is a slice for the conjugation action of G. That is,

$$G \times_{G_I} S_I = V_I .$$

We let $\pi_I \colon V_I \to G/G_I$ denote the projection to the base.

5.2 The basic 3-form on G

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left- and right-invariant Maurer-Cartan forms on G, respectively. The 3-form $\eta \in \Omega^3(G)$ given by[3])

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12}\theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension $\eta_G \in \Omega^3_G(G)$ given by

$$\eta_G(\xi) := \eta - \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi \, .$$

Their cohomology classes represent generators of $H^3(G, \mathbf{Z}) = \mathbf{Z}$ and $H^3_G(G, \mathbf{Z}) = \mathbf{Z}$, respectively. The pull-back of η_G to any conjugacy class $\iota_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow G$ is exact. In fact, let $\omega_{\mathcal{C}} \in \Omega^2(\mathcal{C})^G \subset \Omega^2_G(\mathcal{C})$ be the invariant 2-form given on generating vector fields $\xi_{\mathcal{C}}, \xi'_{\mathcal{C}}$ for $\xi, \xi' \in \mathfrak{g}$ by the formula

$$\omega_{\mathcal{C}}(\xi_{\mathcal{C}}(g), \xi'_{\mathcal{C}}(g)) = \tfrac{1}{2}\xi \cdot (\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi' \, .$$

Then [1, 16]

$$\mathrm{d}_G \, \omega_{\mathcal{C}} + \iota^*_{\mathcal{C}} \eta_G = 0 \, .$$

We will now show that η_G is exact over each of the open subsets V_j. Let $\mathcal{C}_j = q^{-1}(\mu_j) \subset V_j$ be the conjugacy classes corresponding to the vertices.

Lemma 5.1. *The linear retraction*

$$[0,1] \times \mathfrak{A}_j \to \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1-t)\zeta$$

of $\mathfrak{A}_j$ onto the vertex μ_j lifts uniquely to a smooth G-equivariant retraction from V_j onto $\mathcal{C}_j$.

Proof. Recall that the slice S_j is an open neighborhood of $\exp(\mu_j)$ in G_j. Any G_j-equivariant retraction from S_j onto $\exp \mu_j$ extends uniquely to a G-equivariant retraction from $V_j = G \times_{G_j} S_j$ onto $\mathcal{C}_j$. Note that $S'_j = G_j \, . \, (\mathfrak{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$, and that $S'_j \to S_j$, $\zeta \mapsto \exp(\mu_j)\exp(\zeta)$ is a G_j-equivariant diffeomorphism. The linear retraction of S'_j onto the origin gives the desired retraction of S_j. Uniqueness is clear, since the retraction has to preserve $\exp(\mathfrak{A}_j) \subset V_j$, by equivariance.

[3]) For $\mathfrak{g}$-valued forms β_1, β_2, the bracket $[\beta_1, \beta_2]$ denotes the $\mathfrak{g}$-valued form obtained by applying the Lie bracket $[\cdot,\cdot] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to the $\mathfrak{g} \otimes \mathfrak{g}$-valued form $\beta_1 \wedge \beta_2$.

Let

$$\mathbf{h}_j \colon \Omega^p(V_j) \to \Omega^p([0,1] \times V_j) \to \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of $[0,1] \times V_j \to V_j$. It has the property

$$\mathrm{d}_G\, \mathbf{h}_j + \mathbf{h}_j\, \mathrm{d}_G = \mathrm{Id} - \pi_j^* \iota_j^* \tag{5.1}$$

where $\iota_j \colon \mathcal{C}_j \to V_j$ is the inclusion and $\pi_j \colon V_j = G \times_{G_j} S_j \to G/G_j = \mathcal{C}_j$ the projection. Let $(\varpi_j)_G = \mathbf{h}_j \eta_G - \pi_j^* \omega_{\mathcal{C}_j} \in \Omega^2_G(V_j)$, and write $(\varpi_j)_G = \varpi_j - \Psi_j$ where $\varpi_j \in \Omega^2(V_j)$ and $\Psi_j \in \Omega^0(V_j, \mathfrak{g})$.

PROPOSITION 5.2. *The equivariant* 2*-form* $(\varpi_j)_G = \varpi_j - \Psi_j$ *has the following properties.*

(a) $\mathrm{d}_G(\varpi_j)_G = \eta_G$.

(b) *The pull-back of* $(\varpi_j)_G$ *to a conjugacy class* $\mathcal{C} \subset V_j$ *is given by*

$$\iota_{\mathcal{C}}^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_{\mathcal{C}}\,,$$

where $(\omega_{\mathcal{O}})_G$ *is the equivariant symplectic form on the adjoint orbit* $\mathcal{O} = \Psi_j(\mathcal{C})$,

(c) *The pull-back of* Ψ_j *to the conjugacy class* $\mathcal{C}_j$ *vanishes. In fact,* $\Psi_j(\exp \xi) = \xi - \mu_j$ *for all* $\xi \in \mathfrak{A}_j$.

(d) *Over each intersection* $V_{ij} = V_i \cap V_j$, *the difference* $\Psi_i - \Psi_j$ *takes values in the adjoint orbit* $\mathcal{O}_{ij}$ *through* $\mu_j - \mu_i \in \mathfrak{g} \cong \mathfrak{g}^*$. *Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where $p_{ij} \colon V_{ij} \to \mathcal{O}_{ij}$ *is the map defined by* $\Psi_i - \Psi_j$, *and* $(\omega_{\mathcal{O}_{ij}})_G$ *is the equivariant symplectic form on the orbit.*

Proof. (a) holds by construction. (b) follows from the observation that $\iota_{\mathcal{C}}^*(\varpi_j)_G + \omega_{\mathcal{C}}$ is an equivariantly closed 2-form on $\mathcal{C}_j$, with Ψ_j as its moment map. To prove (c) we note that since the retraction is equivariant, we have $\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^* \circ \mathbf{h}_j$ where $(\exp|_{\mathfrak{A}_j})^*$ is pull-back to $\mathfrak{A}_j \subset \mathfrak{t}$ and where $\tilde{\mathbf{h}}_j$ is the homotopy operator for the linear retraction of $\mathfrak{t}$ onto $\{\mu_j\}$. Let $\nu \colon \mathfrak{A}_j \to \mathfrak{t}$ be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* \tfrac{1}{2}(\theta^L + \theta^R) = \tilde{\mathbf{h}}_j \circ \mathrm{d}\nu = \nu - \mu_j\,,$$

proving that $(\exp|_{\mathfrak{A}_j})^* \Psi_j = \nu - \mu_j$. This yields (c), by equivariance. For $\nu \in \mathfrak{A}_{ij}$ we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i \,.$$

By equivariance, it follows that $\Psi_i - \Psi_j$ takes values in the adjoint orbit through $\mu_j - \mu_i$. The difference $\varpi_i - \varpi_j$ vanishes on the maximal torus T, and is therefore determined by its contractions with generating vector fields. Since $\Psi_i - \Psi_j$ is a moment map for $\varpi_i - \varpi_j$, it follows that $\varpi_i - \varpi_j$ equals the pull-back of the symplectic form on $G\,.(\mu_j - \mu_i)$.

5.3 The special unitary group

For the special unitary group $G = \mathrm{SU}(d+1)$, the construction of the basic gerbe simplifies due to the fact that in this case all vertices μ_j of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover $\mathcal{V} = \{V_i,\ i = 0, \ldots, d\}$.

For each weight $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$, let G_μ be its stabilizer for the adjoint action and let $\mathbf{C}_\mu$ the 1-dimensional G_μ-representation with infinitesimal character μ. Let the line bundle $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$ equipped with the unique left-invariant connection ∇. Then L_μ is a G-equivariant pre-quantum line bundle for the orbit $\mathcal{O} = G\,.\,\mu$. That is,

$$\frac{i}{2\pi}\,\mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form and $\Phi_{\mathcal{O}} \colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is the moment map given as inclusion.

In particular, in the case of $\mathrm{SU}(d+1)$ all orbits $\mathcal{O}_{ij} = G\,.(\mu_j - \mu_i)$ carry G-equivariant pre-quantum line bundles. Recall the fibrations $p_{ij} \colon V_{ij} \to \mathcal{O}_{ij}$ defined by $\Psi_i - \Psi_j$, and let

$$L_{ij} = p_{ij}^*\,(L_{\mu_j - \mu_i})\,,$$

equipped with the pull-back connection. For any triple intersection $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$, the tensor product $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$ is the pull-back of the line bundle over G/G_{ijk}, defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of G_{ijk}. It is hence canonically trivial, with $(\delta\nabla)_{ijk}$ the trivial connection. The trivializing section $t_{ijk} = 1$ satisfies $\delta t = 1$ and $(\delta\nabla)t = 0$. Take $(B_j)_G = (\varpi_j)_G$. Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^{L_{ij}})\,.$$

Thus $\mathcal{G} = (\mathcal{V}, L, t)$ is a equivariant gerbe with connection (∇, B). Since

$$\mathrm{d}_G(B_j)_G = \mathrm{d}_G(\varpi_j)_G = \eta_G|_{V_j} ,$$

this is the basic gerbe for $\mathrm{SU}(d+1)$. The transition line bundles L_{ij} may be expressed in terms of eigenspace line bundles, leading to the description of the basic gerbe from the introduction.

REMARK 5.3. This description of the basic gerbe over the special unitary group was found independently by Gawędzki-Reis [13], who also discuss the much more difficult case of quotients of $\mathrm{SU}(d+1)$ by subgroups of the center.

A similar construction works for the group $C_d = \mathrm{Sp}(d)$, the only case besides $A_d = \mathrm{SU}(d+1)$ for which the vertices of the alcove are in the weight lattice. The following table lists, for all simply connected compact simple groups, the smallest integer $k_0 > 0$ such that $k_0\mathfrak{A}$ is a weight lattice polytope[4]). The construction for $\mathrm{SU}(d+1)$ generalizes to describe the k_0-th power of the basic gerbe in all cases.

(5.2)

G	A_d	B_d	C_d	D_d	E_6	E_7	E_8	F_4	G_2
k_0	1	2	1	2	6	12	60	6	2

5.4 THE BASIC GERBE FOR GENERAL SIMPLE, SIMPLY CONNECTED G

The extra difficulty for the groups with $k_0 > 1$ comes from the fact that the pull-back maps $H^3_G(G,\mathbf{Z}) \to H^3_G(\mathcal{C}_j,\mathbf{Z}) \cong H^3_G(V_j,\mathbf{Z})$ may be a non-zero torsion class, in general. In this case the restriction of the basic gerbe to V_j will be non-trivial. Our strategy for the general case is to first construct equivariant bundle gerbes over V_j, and then glue the local data as explained in Section 4.

The centralizers G_g of elements $g \in G$ are always connected [11, Corollary (3.15)] but need not be simply-connected. The conjugacy classes $\mathcal{C}_j = q^{-1}(\mu_j)$ corresponding to the vertices of the alcove are exactly the conjugacy classes of elements for which the centralizer is semi-simple. Since

$$H^3_G(\mathcal{C}_j,\mathbf{Z}) = H^3_G(G/G_j,\mathbf{Z}) = H^3_{G_j}(\mathrm{pt},\mathbf{Z}) ,$$

we see that the torsion problem described above is related to a possibly non-trivial central extension of the centralizers G_j of $\exp(\mu_j)$ by the circle $\mathrm{U}(1)$.

[4]) This information is extracted from the tables in Bourbaki [5]. Letting $\varpi_1,\dots,\varpi_d$ be the fundamental weights, one determines k_0 as the least common multiple of the numbers $\alpha_{max}\cdot\varpi_j$, using the basic inner product defined by $\alpha_{max}\cdot\alpha_{max} = 2$. The number k_0 is equal to the smallest Dynkin index of a representation $G \to \mathrm{SU}(n)$, see [28, p. 128] where the same table appears in a different context.

PROPOSITION 5.4. *Any vertex μ_j of the alcove $\mathfrak{A}$ is in the dual of the co-root lattice for the corresponding centralizer G_j. It hence defines a homomorphism $\varrho_j \in \mathrm{Hom}(\pi_1(G_j), \mathrm{U}(1))$, or equivalently a central extension of G_j by* U(1).

Proof. Let $\widetilde{G}_j$ be the universal cover of G_j. A system of simple roots for $\widetilde{G}_j$ is given by the list of all α_i $(i = 0, \ldots, d)$ with $j \neq i$. The lattice Λ_j is spanned by the corresponding coroots $\check{\alpha}_i$. To show that μ_j is in the dual of the co-root lattice, we have to verify that $\langle \mu_j, \check{\alpha}_i \rangle \in \mathbf{Z}$ for $i \neq j$. For $i \neq 0, j$ this is obvious since $\mu_j(\check{\alpha}_i) = 0$. For $i = 0$, we have $\|\check{\alpha}_0\|^2 = 2$, and therefore $\check{\alpha}_0 = \alpha_0$ and $\mu_j(\check{\alpha}_0) = \alpha_0(\mu_j) = -1$.

Recall that for $i \neq j$, G_{ij} is the centralizer of points $\exp \mu$ with $\mu = t\mu_j + (1-t)\mu_i$ for some $0 < t < 1$. Let $\varrho_{ij} \in \mathrm{Hom}(\pi_1(G_{ij}), \mathrm{U}(1))$ be the quotient of $\pi_1(G_{ij}) \to \pi_1(G_j) \xrightarrow{\varrho_j} \mathrm{U}(1)$ by the homomorphism $\pi_1(G_{ij}) \to \pi_1(G_i) \xrightarrow{\varrho_i} \mathrm{U}(1)$.

LEMMA 5.5. *The difference $\mu_j - \mu_i \in \mathfrak{g}_{ij}$ is fixed under G_{ij}, and $\varrho_{ij} \in \mathrm{Hom}(\pi_1(G_{ij}), \mathrm{U}(1))$ is its image under the exact sequence* (3.2) *for $K = G_{ij}$.*

Proof. Since G_{ij} fixes the curve $g(t) = \exp(t\mu_j + (1-t)\mu_i) = \exp(\mu_i)\exp(t(\mu_j - \mu_i))$, it stabilizes the Lie algebra element $\mu_j - \mu_i$. The second claim is immediate from the definition.

We are now in position to explain our construction of the basic gerbe in the general case. For all $I \subset \{0, \ldots, d\}$ let $X_I \to V_I$ be the G-equivariant principal G_I-bundle,

$$X_I = G \times S_I \to V_I = G \times_{G_I} S_I .$$

X_I is the pull-back of the G_I-bundle $G \to G/G_I$, and in particular carries a G-invariant connection θ_I obtained by pull-back of the unique G-invariant connection on that bundle. For $I \supset J$ there are natural G-equivariant inclusions $f_I^J \colon X_I \to X_J$, and these are compatible as in Section 4. The homomorphisms $\varrho_j \colon \pi_1(G_j) \to \mathrm{U}(1)$ define flat, G-equivariant bundle gerbes $\mathcal{G}_j = (X_j, L_j, t_j)$ over V_j.

The quotient of the two gerbes on V_{ij}, obtained by pulling back $\mathcal{G}_i$, $\mathcal{G}_j$ to X_{ij}, is just the gerbe defined by the homomorphism $\varrho_{ij} \colon \pi_1(G_{ij}) \to \mathrm{U}(1)$. By Lemma 5.5 and Proposition 3.2(b), it follows that this quotient gerbe has a distinguished, equivariant pseudo-line bundle (E_{ij}, s_{ij}) (where E_{ij} is trivial), with connection $\nabla^{E_{ij}}$ induced from the connection θ_{ij}. From the definition of θ_{ij}, it follows that the equivariant error 2-form for this connection is the pull-back of the equivariant symplectic form on the coadjoint orbit through $\mu_j - \mu_i$.

We now modify the bundle gerbe connection by adding the equivariant 2-form $(\varpi_j)_G \in \Omega^2_G(V_j)$ to the gerbe connection. Proposition 5.2(d) shows that the equivariant error 2-form of $\nabla^{E_{ij}}$ with respect to the new gerbe connection vanishes. The other conditions from the gluing construction in §4 are trivially satisfied. Since the equivariant 3-curvature for the new gerbe connection on $\mathcal{G}_j$ is $d_G(\varpi_j)_G = \eta_G|_{V_j}$, we have constructed an equivariant bundle gerbe with connection, with equivariant curvature-form η_G.

REMARK 5.6. For $G = \mathrm{SU}(d+1)$ this construction reduces to the construction in terms of transition line bundles: All L_i, t_i, E_{ij}, u_{ijk} are trivial in this case, hence the entire information on the gerbe resides in the functions $s_{ij}\colon (X_{ij})^{[2]} \to \mathrm{U}(1)$ defined by the differences $\mu_j - \mu_i$. The condition $\delta s_{ij} = 1$ for these functions means that s_{ij} defines a line bundle L_{ij} over V_{ij}, as remarked at the beginning of Section 2.2. The condition $s_{ij}s_{jk}s_{ki} = 1$ over X_{ijk} is the compatibility condition over triple intersections.

6. PRE-QUANTIZATION OF CONJUGACY CLASSES

It is a well-known fact from symplectic geometry that a coadjoint orbit $\mathcal{O} = G.\mu$ through $\mu \in \mathfrak{t}^*_+$ has integral symplectic form, i.e. admits a pre-quantum line bundle, if and only if μ is in the weight lattice Λ^*. The analogous question for conjugacy classes reads: For which $\mu \in \mathfrak{A}$ and $m \in \mathbf{N}$ does the pull-back of the mth power of the basic gerbe $\mathcal{G}^m$ to the conjugacy class $\mathcal{C} = G.\exp(\mu)$ admit a pseudo-line bundle, with $m\omega_{\mathcal{C}}$ as its error 2-form? For any positive integer $m > 0$ let

$$\Lambda^*_m = \Lambda^* \cap m\mathfrak{A}$$

be the set of level m weights. As is well-known [26], the set Λ^*_m parametrizes the positive energy representations of the loop group LG at level m.

THEOREM 6.1. *The restriction of $\mathcal{G}^m$ to a conjugacy class $\mathcal{C}$ admits a pseudo-line bundle $\mathcal{L}$ with connection, with error 2-form $m\omega_{\mathcal{C}}$, if and only if $\mathcal{C} = G.\exp(\mu/m)$ with $\mu \in \Lambda^*_m$. Moreover $\mathcal{L}$ has an equivariant extension in this case, with $m\omega_{\mathcal{C}}$ as its* equivariant *error 2-form* .

Proof. Given a conjugacy class $\mathcal{C} \subset G$, let $\mu \in m\mathfrak{A}$ be the unique point with $g := \exp(\mu/m) \in \mathcal{C}$, and let $K = G_g$ so that $\mathcal{C} = G/K$. Pick an index j with $\mathcal{C} \subset V_j$, and let

$$\nu = m\Psi_j(g) = \mu - m\mu_j .$$

Then

$$G_\mu \subset K \subset G_\nu .$$

Let $\mathcal{O}_\mu, \mathcal{O}_\nu \subset \mathfrak{g}$ denote the adjoint orbits of μ, ν, and $(\omega_\mu)_G$, $(\omega_\nu)_G$ their equivariant symplectic forms. The pull-back $\iota_{\mathcal{C}}^* \mathcal{G}^m$ is the gerbe over G/K defined as in Section 3 by the homomorphism $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$, given as a composition

$$\pi_1(K) \to \pi_1(G_j) \to \mathrm{U}(1) ,$$

where the fist map is push-forward under the inclusion $K \hookrightarrow G_j$, and the second map is the homomorphism defined by the element $m\mu_j \in \mathfrak{t}$ for G_j.

Suppose now that $\mu \in \Lambda_m^*$. Then $m\mu_j$ equals $-\nu$ up to a weight lattice vector, which means that ϱ is the image of $-\nu \in (\mathfrak{k}^*)^K$ in the exact sequence (3.2). Hence, Proposition 3.2 says that we we obtain an equivariant pseudo-line bundle for $\iota_{\mathcal{C}}^* \mathcal{G}^m$, with equivariant error 2-form

$$\Psi_j^*(\omega_\nu)_G - m\, \iota_{\mathcal{C}}^*(\varpi_j)_G = m\, \omega_{\mathcal{C}} .$$

Here we have used part (b) of Proposition 5.2.

Conversely, suppose that $\mathcal{G}^m|_{\mathcal{C}}$ admits a pseudo-line bundle with error 2-form $m\,\omega_{\mathcal{C}}$. Consider the pull-back of $\mathcal{G}$ under the exponential map $\exp\colon \mathfrak{g} \to G$. The pull-back $\exp^* \eta \in \Omega^3(\mathfrak{g})$ is exact, and the homotopy operator for the linear retraction of $\mathfrak{g}$ to the origin defines a 2-form $\varpi \in \Omega^2(\mathfrak{g})$ with $\mathrm{d}\varpi = \exp^* \eta$. As in Proposition 5.2, one shows that for any adjoint orbit $\mathcal{O} \subset \mathfrak{g}$, with $\exp \mathcal{O} = \mathcal{C}$,

$$\iota_{\mathcal{O}}^* \varpi = \exp^* \omega_{\mathcal{C}} - \omega_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form on $\mathcal{O}$. In particular this applies to $\mathcal{O} = \mathcal{O}_{\mu/m}$. Choose a pseudo-line bundle for $\exp^* \mathcal{G}$ with error 2-form $-\varpi$. We then have two pseudo-line bundles for $\exp^* \mathcal{G}^m|_{\mathcal{O}}$ obtained by restricting the mth power of the pseudo-line bundle for $\exp^* \mathcal{G}$ or by pulling back the pseudo-line bundle for $\mathcal{C}$. Their quotient is a line bundle over $\mathcal{O}$, with curvature the difference of the error 2-forms:

$$m(\exp^* \omega_{\mathcal{C}} - \iota_{\mathcal{O}_\mu}^* \varpi) = m\omega_{\mathcal{O}} .$$

Thus $m(\mu/m) = \mu$ must be in the weight lattice.

REMARK 6.2. Z. Shahbazi has proved that if $\mathcal{G}$ is a gerbe with connection over a manifold M, with curvature 3-form η, and $\Phi\colon N \to M$ is a map with $\Phi^*\eta + d\omega = 0$, then the pull-back gerbe $\Phi^*\mathcal{G}$ admits a pseudo-line bundle, with ω as its error 2-form, if and only if the pair (η, ω) defines an integral element of the relative de Rham cohomology $H^3(\Phi, \mathbf{R})$. This means that for any smooth 2-cycle $S \subset N$, and any smooth 3-chain $B \subset M$ with boundary $\Phi(S)$, one must have $\int_B \eta - \int_S \omega \in \mathbf{Z}$. The particular case where the target of Φ is a Lie group G is relevant for the pre-quantization of group-valued moment maps [1].

APPENDIX A. PROOF OF LEMMA 4.4

In this Appendix we prove Lemma 4.4, concerning the construction of a certain cover U_I of M from a given cover V_j. Write $M = \coprod_I A_I$ where

$$A_I = \bigcap_{i\in I} V_i \backslash \bigcup_{j\notin I} V_j .$$

Notice that $\bar{A}_I \subset \bigcup_{J\subset I} A_J$. By induction on the cardinality $k = |I|$ we will construct open sets $U_J \subset V_J$, having the following properties:

(a) the closure $\bar{U}_I$ does not meet $\bar{U}_J$ for $|J| \leq |I|$ unless $J \subset I$,

(b) each $\bar{A}_I$ is contained in the union of U_J with $J \subset I$.

The induction starts at $k = 0$, taking $U_\varnothing = \varnothing$. Suppose we have constructed open sets U_I with $\bar{U}_I \subset V_I$ for $|I| < k$, such that the properties (a), (b) hold for all $|I| < k$. For $|I| = k$ consider the subsets

$$B_I := A_I \backslash \Big(\bigcup_{J\subset I, |J|<k} U_J \Big) .$$

Note that (unlike A_I) the set B_I is closed. B_I does not meet $\bar{A}_J$ unless $I \subset J$, and it also does not meet $\bar{U}_J$ for $|J| < k$ unless $J \subset I$. That is, B_I is disjoint from

$$C_I := \bigcup_{J\not\subset I, |J|<k} \bar{U}_J \cup \bigcup_{K\not\supset I} \bar{A}_K .$$

Choose open sets U_I for $|I| = k$ with $B_I \subset U_I \subset \bar{U}_I \subset M\backslash C_I$, and such that the closures of the sets U_I for distinct I with $|I| = k$ are disjoint. The new collection of subsets will satisfy the properties (a), (b) for $|I| \leq k$. We next show that $V_i' = M\backslash \bigcup_{J\not\ni i} \bar{U}_J$ is a cover of M. Write $M = \coprod_I D_I$ with $D_I = \bar{U}_I \backslash \bigcup_{|J|<|I|} \bar{U}_J$. Then $D_I \cap \bar{U}_J = \varnothing$ unless $I \subset J$, so D_I is contained in each V_i' with $i \in I$. In particular $\bigcup_i V_i' = M$. Finally $\overline{V_i'} \subset \bigcup_{I \ni i} \overline{U}_I \subset V_i$. This completes the proof of Lemma 4.4. Note that if the V_i were invariant under an action of a compact group G, the U_I could be taken G-invariant also.

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