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5.2 THE BASIC 3-FORM ON G

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left- and right-invariant Maurer-Cartan forms on G, respectively. The 3-form $\eta \in \Omega^3(G)$ given by[3])

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12}\theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension $\eta_G \in \Omega^3_G(G)$ given by

$$\eta_G(\xi) := \eta - \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi .$$

Their cohomology classes represent generators of $H^3(G, \mathbf{Z}) = \mathbf{Z}$ and $H^3_G(G, \mathbf{Z}) = \mathbf{Z}$, respectively. The pull-back of η_G to any conjugacy class $\iota_\mathcal{C} \colon \mathcal{C} \hookrightarrow G$ is exact. In fact, let $\omega_\mathcal{C} \in \Omega^2(\mathcal{C})^G \subset \Omega^2_G(\mathcal{C})$ be the invariant 2-form given on generating vector fields $\xi_\mathcal{C}, \xi'_\mathcal{C}$ for $\xi, \xi' \in \mathfrak{g}$ by the formula

$$\omega_\mathcal{C}(\xi_\mathcal{C}(g), \xi'_\mathcal{C}(g)) = \tfrac{1}{2}\xi \cdot (\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi' .$$

Then [1, 16]

$$\mathrm{d}_G \, \omega_\mathcal{C} + \iota^*_\mathcal{C} \eta_G = 0 .$$

We will now show that η_G is exact over each of the open subsets V_j. Let $\mathcal{C}_j = q^{-1}(\mu_j) \subset V_j$ be the conjugacy classes corresponding to the vertices.

LEMMA 5.1. *The linear retraction*

$$[0, 1] \times \mathfrak{A}_j \to \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1 - t)\,\zeta$$

of $\mathfrak{A}_j$ onto the vertex μ_j lifts uniquely to a smooth G-equivariant retraction from V_j onto $\mathcal{C}_j$.

Proof. Recall that the slice S_j is an open neighborhood of $\exp(\mu_j)$ in G_j. Any G_j-equivariant retraction from S_j onto $\exp \mu_j$ extends uniquely to a G-equivariant retraction from $V_j = G \times_{G_j} S_j$ onto $\mathcal{C}_j$. Note that $S'_j = G_j \,.\, (\mathfrak{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$, and that $S'_j \to S_j$, $\zeta \mapsto \exp(\mu_j)\exp(\zeta)$ is a G_j-equivariant diffeomorphism. The linear retraction of S'_j onto the origin gives the desired retraction of S_j. Uniqueness is clear, since the retraction has to preserve $\exp(\mathfrak{A}_j) \subset V_j$, by equivariance.

[3]) For $\mathfrak{g}$-valued forms β_1, β_2, the bracket $[\beta_1, \beta_2]$ denotes the $\mathfrak{g}$-valued form obtained by applying the Lie bracket $[\cdot,\cdot] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to the $\mathfrak{g} \otimes \mathfrak{g}$-valued form $\beta_1 \wedge \beta_2$.

Let

$$\mathbf{h}_j\colon \Omega^p(V_j) \to \Omega^p([0,1]\times V_j) \to \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of $[0,1]\times V_j \to V_j$. It has the property

$$\mathrm{d}_G\,\mathbf{h}_j + \mathbf{h}_j\,\mathrm{d}_G = \mathrm{Id} - \pi_j^*\iota_j^* \tag{5.1}$$

where $\iota_j\colon \mathcal{C}_j \to V_j$ is the inclusion and $\pi_j\colon V_j = G\times_{G_j} S_j \to G/G_j = \mathcal{C}_j$ the projection. Let $(\varpi_j)_G = \mathbf{h}_j\eta_G - \pi_j^*\omega_{\mathcal{C}_j} \in \Omega^2_G(V_j)$, and write $(\varpi_j)_G = \varpi_j - \Psi_j$ where $\varpi_j \in \Omega^2(V_j)$ and $\Psi_j \in \Omega^0(V_j,\mathfrak{g})$.

PROPOSITION 5.2. *The equivariant* 2*-form* $(\varpi_j)_G = \varpi_j - \Psi_j$ *has the following properties.*

(a) $\mathrm{d}_G(\varpi_j)_G = \eta_G$.

(b) *The pull-back of* $(\varpi_j)_G$ *to a conjugacy class* $\mathcal{C}\subset V_j$ *is given by*

$$\iota_{\mathcal{C}}^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_{\mathcal{C}}\,,$$

where $(\omega_{\mathcal{O}})_G$ *is the equivariant symplectic form on the adjoint orbit* $\mathcal{O} = \Psi_j(\mathcal{C})$,

(c) *The pull-back of* Ψ_j *to the conjugacy class* $\mathcal{C}_j$ *vanishes. In fact,* $\Psi_j(\exp\xi) = \xi - \mu_j$ *for all* $\xi \in \mathfrak{A}_j$.

(d) *Over each intersection* $V_{ij} = V_i \cap V_j$, *the difference* $\Psi_i - \Psi_j$ *takes values in the adjoint orbit* $\mathcal{O}_{ij}$ *through* $\mu_j - \mu_i \in \mathfrak{g} \cong \mathfrak{g}^*$. *Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where $p_{ij}\colon V_{ij} \to \mathcal{O}_{ij}$ *is the map defined by* $\Psi_i - \Psi_j$, *and* $(\omega_{\mathcal{O}_{ij}})_G$ *is the equivariant symplectic form on the orbit.*

Proof. (a) holds by construction. (b) follows from the observation that $\iota_{\mathcal{C}}^*(\varpi_j)_G + \omega_{\mathcal{C}}$ is an equivariantly closed 2-form on $\mathcal{C}_j$, with Ψ_j as its moment map. To prove (c) we note that since the retraction is equivariant, we have $\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^* \circ \mathbf{h}_j$ where $(\exp|_{\mathfrak{A}_j})^*$ is pull-back to $\mathfrak{A}_j \subset \mathfrak{t}$ and where $\tilde{\mathbf{h}}_j$ is the homotopy operator for the linear retraction of $\mathfrak{t}$ onto $\{\mu_j\}$. Let $\nu\colon \mathfrak{A}_j \to \mathfrak{t}$ be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* \tfrac{1}{2}(\theta^L + \theta^R) = \tilde{\mathbf{h}}_j \circ \mathrm{d}\nu = \nu - \mu_j\,,$$

proving that $(\exp|_{\mathfrak{A}_j})^*\Psi_j = \nu - \mu_j$. This yields (c), by equivariance. For $\nu \in \mathfrak{A}_{ij}$ we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i\,.$$

By equivariance, it follows that $\Psi_i - \Psi_j$ takes values in the adjoint orbit through $\mu_j - \mu_i$. The difference $\varpi_i - \varpi_j$ vanishes on the maximal torus T, and is therefore determined by its contractions with generating vector fields. Since $\Psi_i - \Psi_j$ is a moment map for $\varpi_i - \varpi_j$, it follows that $\varpi_i - \varpi_j$ equals the pull-back of the symplectic form on $G\,.(\mu_j - \mu_i)$.

5.3 The special unitary group

For the special unitary group $G = \mathrm{SU}(d+1)$, the construction of the basic gerbe simplifies due to the fact that in this case all vertices μ_j of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover $\mathcal{V} = \{V_i,\ i = 0, \ldots, d\}$.

For each weight $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$, let G_μ be its stabilizer for the adjoint action and let $\mathbf{C}_\mu$ the 1-dimensional G_μ-representation with infinitesimal character μ. Let the line bundle $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$ equipped with the unique left-invariant connection ∇. Then L_μ is a G-equivariant pre-quantum line bundle for the orbit $\mathcal{O} = G\,.\,\mu$. That is,

$$\frac{i}{2\pi}\,\mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where $\omega_{\mathcal{O}}$ is the symplectic form and $\Phi_{\mathcal{O}} \colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is the moment map given as inclusion.

In particular, in the case of $\mathrm{SU}(d+1)$ all orbits $\mathcal{O}_{ij} = G\,.(\mu_j - \mu_i)$ carry G-equivariant pre-quantum line bundles. Recall the fibrations $p_{ij} \colon V_{ij} \to \mathcal{O}_{ij}$ defined by $\Psi_i - \Psi_j$, and let

$$L_{ij} = p_{ij}^*\,(L_{\mu_j - \mu_i})\,,$$

equipped with the pull-back connection. For any triple intersection $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$, the tensor product $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$ is the pull-back of the line bundle over G/G_{ijk}, defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of G_{ijk}. It is hence canonically trivial, with $(\delta\nabla)_{ijk}$ the trivial connection. The trivializing section $t_{ijk} = 1$ satisfies $\delta t = 1$ and $(\delta\nabla) t = 0$. Take $(B_j)_G = (\varpi_j)_G$. Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^{L_{ij}})\,.$$

Thus $\mathcal{G} = (\mathcal{V}, L, t)$ is a equivariant gerbe with connection (∇, B). Since