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a distinguished, equivariant pseudo-line bundle  $(E_{ij}, s_{ij})$  (where  $E_{ij}$  is trivial), with connection  $\nabla^{E_{ij}}$  induced from the connection  $\theta_{ij}$ . From the definition of  $\theta_{ij}$ , it follows that the equivariant error 2-form for this connection is the pull-back of the equivariant symplectic form on the coadjoint orbit through  $\mu_j - \mu_i$ .

We now modify the bundle gerbe connection by adding the equivariant 2-form  $(\varpi_j)_G \in \Omega_G^2(V_j)$  to the gerbe connection. Proposition 5.2(d) shows that the equivariant error 2-form of  $\nabla^{E_{ij}}$  with respect to the new gerbe connection vanishes. The other conditions from the gluing construction in §4 are trivially satisfied. Since the equivariant 3-curvature for the new gerbe connection on  $\mathcal{G}_j$  is  $d_G(\varpi_j)_G = \eta_G|_{V_j}$ , we have constructed an equivariant bundle gerbe with connection, with equivariant curvature-form  $\eta_G$ .

REMARK 5.6. For  $G = \text{SU}(d + 1)$  this construction reduces to the construction in terms of transition line bundles: All  $L_i, t_i, E_{ij}, u_{ijk}$  are trivial in this case, hence the entire information on the gerbe resides in the functions  $s_{ij}: (X_{ij})^{[2]} \rightarrow \text{U}(1)$  defined by the differences  $\mu_j - \mu_i$ . The condition  $\delta s_{ij} = 1$  for these functions means that  $s_{ij}$  defines a line bundle  $L_{ij}$  over  $V_{ij}$ , as remarked at the beginning of Section 2.2. The condition  $s_{ij}s_{jk}s_{ki} = 1$  over  $X_{ijk}$  is the compatibility condition over triple intersections.

### 6. PRE-QUANTIZATION OF CONJUGACY CLASSES

It is a well-known fact from symplectic geometry that a coadjoint orbit  $\mathcal{O} = G \cdot \mu$  through  $\mu \in \mathfrak{t}_+^*$  has integral symplectic form, i.e. admits a pre-quantum line bundle, if and only if  $\mu$  is in the weight lattice  $\Lambda^*$ . The analogous question for conjugacy classes reads: For which  $\mu \in \mathfrak{A}$  and  $m \in \mathbf{N}$  does the pull-back of the  $m$ th power of the basic gerbe  $\mathcal{G}^m$  to the conjugacy class  $\mathcal{C} = G \cdot \exp(\mu)$  admit a pseudo-line bundle, with  $m\omega_{\mathcal{C}}$  as its error 2-form? For any positive integer  $m > 0$  let

$$\Lambda_m^* = \Lambda^* \cap m\mathfrak{A}$$

be the set of level  $m$  weights. As is well-known [26], the set  $\Lambda_m^*$  parametrizes the positive energy representations of the loop group  $LG$  at level  $m$ .

THEOREM 6.1. *The restriction of  $\mathcal{G}^m$  to a conjugacy class  $\mathcal{C}$  admits a pseudo-line bundle  $\mathcal{L}$  with connection, with error 2-form  $m\omega_{\mathcal{C}}$ , if and only if  $\mathcal{C} = G \cdot \exp(\mu/m)$  with  $\mu \in \Lambda_m^*$ . Moreover  $\mathcal{L}$  has an equivariant extension in this case, with  $m\omega_{\mathcal{C}}$  as its equivariant error 2-form.*

*Proof.* Given a conjugacy class  $\mathcal{C} \subset G$ , let  $\mu \in m\mathfrak{A}$  be the unique point with  $g := \exp(\mu/m) \in \mathcal{C}$ , and let  $K = G_g$  so that  $\mathcal{C} = G/K$ . Pick an index  $j$  with  $\mathcal{C} \subset V_j$ , and let

$$\nu = m\Psi_j(g) = \mu - m\mu_j.$$

Then

$$G_\mu \subset K \subset G_\nu.$$

Let  $\mathcal{O}_\mu, \mathcal{O}_\nu \subset \mathfrak{g}$  denote the adjoint orbits of  $\mu, \nu$ , and  $(\omega_\mu)_G, (\omega_\nu)_G$  their equivariant symplectic forms. The pull-back  $\iota_{\mathcal{C}}^* \mathcal{G}^m$  is the gerbe over  $G/K$  defined as in Section 3 by the homomorphism  $\varrho \in \text{Hom}(\pi_1(K), \text{U}(1))$ , given as a composition

$$\pi_1(K) \rightarrow \pi_1(G_j) \rightarrow \text{U}(1),$$

where the first map is push-forward under the inclusion  $K \hookrightarrow G_j$ , and the second map is the homomorphism defined by the element  $m\mu_j \in \mathfrak{t}$  for  $G_j$ .

Suppose now that  $\mu \in \Lambda_m^*$ . Then  $m\mu_j$  equals  $-\nu$  up to a weight lattice vector, which means that  $\varrho$  is the image of  $-\nu \in (\mathfrak{k}^*)^K$  in the exact sequence (3.2). Hence, Proposition 3.2 says that we obtain an equivariant pseudo-line bundle for  $\iota_{\mathcal{C}}^* \mathcal{G}^m$ , with equivariant error 2-form

$$\Psi_j^*(\omega_\nu)_G - m\iota_{\mathcal{C}}^*(\varpi_j)_G = m\omega_{\mathcal{C}}.$$

Here we have used part (b) of Proposition 5.2.

Conversely, suppose that  $\mathcal{G}^m|_{\mathcal{C}}$  admits a pseudo-line bundle with error 2-form  $m\omega_{\mathcal{C}}$ . Consider the pull-back of  $\mathcal{G}$  under the exponential map  $\exp: \mathfrak{g} \rightarrow G$ . The pull-back  $\exp^* \eta \in \Omega^3(\mathfrak{g})$  is exact, and the homotopy operator for the linear retraction of  $\mathfrak{g}$  to the origin defines a 2-form  $\varpi \in \Omega^2(\mathfrak{g})$  with  $d\varpi = \exp^* \eta$ . As in Proposition 5.2, one shows that for any adjoint orbit  $\mathcal{O} \subset \mathfrak{g}$ , with  $\exp \mathcal{O} = \mathcal{C}$ ,

$$\iota_{\mathcal{O}}^* \varpi = \exp^* \omega_{\mathcal{C}} - \omega_{\mathcal{O}}$$

where  $\omega_{\mathcal{O}}$  is the symplectic form on  $\mathcal{O}$ . In particular this applies to  $\mathcal{O} = \mathcal{O}_{\mu/m}$ . Choose a pseudo-line bundle for  $\exp^* \mathcal{G}$  with error 2-form  $-\varpi$ . We then have two pseudo-line bundles for  $\exp^* \mathcal{G}^m|_{\mathcal{O}}$  obtained by restricting the  $m$ th power of the pseudo-line bundle for  $\exp^* \mathcal{G}$  or by pulling back the pseudo-line bundle for  $\mathcal{C}$ . Their quotient is a line bundle over  $\mathcal{O}$ , with curvature the difference of the error 2-forms:

$$m(\exp^* \omega_{\mathcal{C}} - \iota_{\mathcal{O}_\mu}^* \varpi) = m\omega_{\mathcal{O}}.$$

Thus  $m(\mu/m) = \mu$  must be in the weight lattice.

REMARK 6.2. Z. Shahbazi has proved that if  $\mathcal{G}$  is a gerbe with connection over a manifold  $M$ , with curvature 3-form  $\eta$ , and  $\Phi: N \rightarrow M$  is a map with  $\Phi^*\eta + d\omega = 0$ , then the pull-back gerbe  $\Phi^*\mathcal{G}$  admits a pseudo-line bundle, with  $\omega$  as its error 2-form, if and only if the pair  $(\eta, \omega)$  defines an integral element of the relative de Rham cohomology  $H^3(\Phi, \mathbf{R})$ . This means that for any smooth 2-cycle  $S \subset N$ , and any smooth 3-chain  $B \subset M$  with boundary  $\Phi(S)$ , one must have  $\int_B \eta - \int_S \omega \in \mathbf{Z}$ . The particular case where the target of  $\Phi$  is a Lie group  $G$  is relevant for the pre-quantization of group-valued moment maps [1].

APPENDIX A. PROOF OF LEMMA 4.4

In this Appendix we prove Lemma 4.4, concerning the construction of a certain cover  $U_I$  of  $M$  from a given cover  $V_j$ . Write  $M = \coprod_I A_I$  where

$$A_I = \bigcap_{i \in I} V_i \setminus \bigcup_{j \notin I} V_j.$$

Notice that  $\bar{A}_I \subset \bigcup_{J \subset I} A_J$ . By induction on the cardinality  $k = |I|$  we will construct open sets  $U_I \subset V_I$ , having the following properties:

- (a) the closure  $\bar{U}_I$  does not meet  $\bar{U}_J$  for  $|J| \leq |I|$  unless  $J \subset I$ ,
- (b) each  $\bar{A}_I$  is contained in the union of  $U_J$  with  $J \subset I$ .

The induction starts at  $k = 0$ , taking  $U_\emptyset = \emptyset$ . Suppose we have constructed open sets  $U_I$  with  $\bar{U}_I \subset V_I$  for  $|I| < k$ , such that the properties (a), (b) hold for all  $|I| < k$ . For  $|I| = k$  consider the subsets

$$B_I := A_I \setminus \left( \bigcup_{J \subset I, |J| < k} U_J \right).$$

Note that (unlike  $A_I$ ) the set  $B_I$  is closed.  $B_I$  does not meet  $\bar{A}_J$  unless  $I \subset J$ , and it also does not meet  $\bar{U}_J$  for  $|J| < k$  unless  $J \subset I$ . That is,  $B_I$  is disjoint from

$$C_I := \bigcup_{J \not\subset I, |J| < k} \bar{U}_J \cup \bigcup_{K \not\subset I} \bar{A}_K.$$

Choose open sets  $U_I$  for  $|I| = k$  with  $B_I \subset U_I \subset \bar{U}_I \subset M \setminus C_I$ , and such that the closures of the sets  $U_I$  for distinct  $I$  with  $|I| = k$  are disjoint. The new collection of subsets will satisfy the properties (a), (b) for  $|I| \leq k$ . We next show that  $V'_i = M \setminus \bigcup_{J \not\supset i} \bar{U}_J$  is a cover of  $M$ . Write  $M = \coprod_I D_I$  with  $D_I = \bar{U}_I \setminus \bigcup_{|J| < |I|} \bar{U}_J$ . Then  $D_I \cap \bar{U}_J = \emptyset$  unless  $I \subset J$ , so  $D_I$  is contained