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REMARK 6.2. Z. Shahbazi has proved that if $\mathcal{G}$ is a gerbe with connection over a manifold M, with curvature 3-form η, and $\Phi\colon N \to M$ is a map with $\Phi^*\eta + \mathrm{d}\omega = 0$, then the pull-back gerbe $\Phi^*\mathcal{G}$ admits a pseudo-line bundle, with ω as its error 2-form, if and only if the pair (η, ω) defines an integral element of the relative de Rham cohomology $H^3(\Phi, \mathbf{R})$. This means that for any smooth 2-cycle $S \subset N$, and any smooth 3-chain $B \subset M$ with boundary $\Phi(S)$, one must have $\int_B \eta - \int_S \omega \in \mathbf{Z}$. The particular case where the target of Φ is a Lie group G is relevant for the pre-quantization of group-valued moment maps [1].

APPENDIX A. PROOF OF LEMMA 4.4

In this Appendix we prove Lemma 4.4, concerning the construction of a certain cover U_I of M from a given cover V_j. Write $M = \coprod_I A_I$ where

$$A_I = \bigcap_{i \in I} V_i \setminus \bigcup_{j \notin I} V_j .$$

Notice that $\bar{A}_I \subset \bigcup_{J \subset I} A_J$. By induction on the cardinality $k = |I|$ we will construct open sets $U_J \subset V_J$, having the following properties:

(a) the closure $\bar{U}_I$ does not meet $\bar{U}_J$ for $|J| \leq |I|$ unless $J \subset I$,

(b) each $\bar{A}_I$ is contained in the union of U_J with $J \subset I$.

The induction starts at $k = 0$, taking $U_\varnothing = \varnothing$. Suppose we have constructed open sets U_I with $\bar{U}_I \subset V_I$ for $|I| < k$, such that the properties (a), (b) hold for all $|I| < k$. For $|I| = k$ consider the subsets

$$B_I := A_I \setminus \Big(\bigcup_{J \subset I, |J| < k} U_J \Big) .$$

Note that (unlike A_I) the set B_I is closed. B_I does not meet $\bar{A}_J$ unless $I \subset J$, and it also does not meet $\bar{U}_J$ for $|J| < k$ unless $J \subset I$. That is, B_I is disjoint from

$$C_I := \bigcup_{J \not\subset I, |J| < k} \bar{U}_J \cup \bigcup_{K \not\supset I} \bar{A}_K .$$

Choose open sets U_I for $|I| = k$ with $B_I \subset U_I \subset \bar{U}_I \subset M \setminus C_I$, and such that the closures of the sets U_I for distinct I with $|I| = k$ are disjoint. The new collection of subsets will satisfy the properties (a), (b) for $|I| \leq k$. We next show that $V_i' = M \setminus \bigcup_{J \not\ni i} \bar{U}_J$ is a cover of M. Write $M = \coprod_I D_I$ with $D_I = \bar{U}_I \setminus \bigcup_{|J| < |I|} \bar{U}_J$. Then $D_I \cap \bar{U}_J = \varnothing$ unless $I \subset J$, so D_I is contained in each V_i' with $i \in I$. In particular $\bigcup_i V_i' = M$. Finally $\overline{V_i'} \subset \bigcup_{I \ni i} \overline{U}_I \subset V_i$. This completes the proof of Lemma 4.4. Note that if the V_i were invariant under an action of a compact group G, the U_I could be taken G-invariant also.

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