Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE CLASSIFICATION OF RATIONAL KNOTS
Autor: Kauffman, Louis H. / Lambropoulou, Sofia
Kapitel: 5. On connectivity
DOI: https://doi.org/10.5169/seals-66693

the equation $qx \equiv -1 \mod p$ and both q and q' are between 1 and $p-1$. Since this equation has a unique solution in this range, we conclude that $q = q'$. It follows at once that the continued fraction sequence for p/q is symmetric.

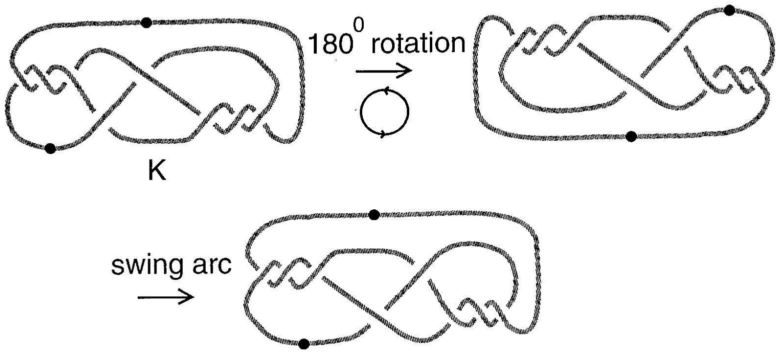

FIGURE 29
An achiral rational link

It is then easy to see that the corresponding rational knot or link $K = N(T)$ is equivalent to its mirror image. One rotates K by 180° in the plane and swings an arc, as Figure 29 illustrates. The point is that the crossings of the second row of the tangle T, that are seemingly crossings of opposite type than the crossings of the upper row, become after the turn crossings of the upper row, and so the types of crossings are switched. This completes the proof. □

5. ON CONNECTIVITY

We shall now introduce the notion of *connectivity* and we shall relate it to the fraction of unoriented rational tangles. We shall say that an unoriented rational tangle has *connectivity type* [0] if the NW end arc is connected to the NE end arc and the SW end arc is connected to the SE end arc. These are the same connections as in the tangle [0]. Similarly, we say that the tangle has *connectivity type* $[\infty]$ *or* [1] if the end arc connections are the same as in the tangles $[\infty]$ and $[+1]$ (or equivalently $[-1]$) respectively. The basic connectivity patterns of rational tangles are exemplified by the tangles [0], $[\infty]$ and $[+1]$. We can represent them iconically by

$$[0] = \asymp$$

$$[\infty] = ><$$

$$[1] = \chi$$

For connectivity we are only concerned with the connection patterns of the four end arcs. Thus $[n]$ has connectivity χ whenever n is odd, and connectivity $\asymp$ whenever n is even.

Note that connectivity type [0] yields two-component rational links, whilst type [1] or $[\infty]$ yields one-component rational links. Also, adding a bottom twist to a rational tangle of connectivity type [0] will not change the connectivity type of the tangle, while adding a bottom twist to a rational tangle of connectivity type $[\infty]$ will switch the connectivity type to [1] and vice versa.

We need to keep an accounting of the connectivity of rational tangles in relation to the parity of the numerators and denominators of their fractions. We adopt the following notation: e stands for *even* and o for *odd*. The *parity of a fraction* p/q is defined to be the ratio of the parities (e or o) of its numerator and denominator p and q. Thus the fraction $2/3$ is of parity e/o. The tangle [0] has fraction $0 = 0/1$, thus parity e/o. The tangle $[\infty]$ has formal fraction $\infty = 1/0$, thus parity o/e. The tangle $[+1]$ has fraction $1 = 1/1$, thus parity o/o, and the tangle $[-1]$ has fraction $-1 = -1/1$, thus parity o/o. We then have the following result.

THEOREM 6. *A rational tangle T has connectivity type $\asymp$ if and only if its fraction has parity e/o. T has connectivity type $><$ if and only if its fraction has parity o/e. Finally, T has connectivity type χ if and only if its fraction has parity o/o.*

Proof. Since $F([0]) = 0/1$, $F([\pm1]) = \pm1/1$ and $F([\infty]) = 1/0$, the theorem is true for these elementary tangles. It remains to show by induction that it is true for any rational tangle T. Note how connectivity type behaves under the addition and product of tangles:

$$\chi + \chi = \asymp$$
$$\chi + \asymp = \chi$$
$$\asymp + \asymp = \asymp$$
$$\chi + >< = ><$$
$$\asymp + >< = ><$$
$$>< + >< = ><>< = \delta ><$$

$$\chi * \chi = ><$$
$$\chi * \asymp = \asymp$$
$$\asymp * \asymp = \delta \asymp$$
$$\chi * >< = \chi$$
$$\asymp * >< = \asymp$$
$$>< * >< = ><$$

The symbol δ stands for the value of a loop formed. Now any rational tangle can be built from [0] or [∞] by successive addition or multiplication with [± 1]. Thus, from the point of view of connectivity, it suffices to show that $[T]+[\pm 1]$ and $[T]*[\pm 1]$ satisfy the theorem whenever $[T]$ satisfies the theorem. This is checked by comparing the connectivity identities above with the parity of the fractions. For example, in the case

$$\chi + \chi = \asymp \quad \text{we have } o/o + o/o = e/o$$

exactly in accordance with the connectivity identity. The other cases correspond as well, and this proves the theorem by induction. □

COROLLARY 1. *For a rational tangle T the link $N(T)$ has two components if and only if T has fraction $F(T)$ of parity e/o.*

Proof. By the Theorem we have $F(T)$ has parity e/o if and only if T has connectivity type $\asymp$. It follows at once that $N(T)$ has two components. □

Another useful fact is that the components of a rational link are individually unknotted embeddings in three dimensional space. To see this, observe that upon removing one strand of a rational tangle, the other strand is an unknotted arc.

6. THE ORIENTED CASE

Oriented rational knots and links are numerator (and denominator) closures of oriented rational tangles. Rational tangles are oriented by choosing an orientation for each strand of the tangle. Two oriented rational tangles are *isotopic* if they are isotopic as unoriented tangles via an isotopy that carries the orientation of one tangle to the orientation of the other. Since the end arcs of a tangle are fixed during a tangle isotopy, this means that isotopic tangles must