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Autor: Etingof, Pavel / Strickland, Elisabetta

Kapitel: 1.4 FURTHER PROPERTIES OF X_m

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W_z-invariant, we deduce that $p(x)-p(sx)=0$, so that in this case $p(x)-p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z)\neq 0$. Indeed, for a reflection s, α_s vanishes exactly on the fixed points of s, so that $\prod_{s\in\Sigma, sz\neq z}\alpha_s(z)^{2m_s+1}\neq 0$. Also for all $w\in W_z$ $f(wz)=f(z)\neq 0$. On the other hand, it is clear that $p(y)=0$. □

EXAMPLE 1.5. Take $W=\mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i}\mid i\geq 0\}\cup\{x^{2i+1}\mid i\geq m\}$. From this we deduce that setting $z=x^2$ and $y=x^{2m+1}$, $Q_m=\mathbf{C}[y,z]/(y^2-z^{2m+1})=\mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2=z^{2m+1}$. The map $\pi\colon \mathbf{C}\to K$ is given by $\pi(t)=(t^{2m+1},t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over $\mathbf{C}$ and $A=\mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1,\dots,f_n\in\mathbf{C}[X]$ which are algebraically independent over $\mathbf{C}$ and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist $f_1,\dots,f_n$ as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1,\dots,f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any $f_1,\dots,f_n$ which are algebraically independent over $\mathbf{C}$ and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$, we have that A is a locally free $\mathbf{C}[f_1,\dots,f_n]$-module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). *Q_m is Cohen-Macaulay.*

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). *Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module.*

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the $\mathbf{C}[\mathfrak{h}]^W$-module Q_m can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category $\mathcal{O}$. On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra $\mathbf{C}[\mathfrak{h}]^W$.

1.5 The Poincaré series of Q_m

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r ,$$

where $Q_m[r]$ denotes the graded component of Q_m of degree r. For every irreducible representation $\tau \in \widehat{W}$, define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \mathrm{Hom}_W(\tau, \mathbf{C}[\mathfrak{h}][r]) t^r .$$

Consider the element in the group ring $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s) .$$

The W-invariance of m implies that μ_m lies in the center of $\mathbf{Z}[W]$. Hence it is clear that μ_m acts as a scalar, $\xi_m(\tau)$, on τ. Let d_τ be the degree of τ.

Lemma 1.9. *The scalar $\xi_m(\tau)$ is an integer.*

Proof. $\mathbf{Z}[W]$ and hence also its center, is a finite $\mathbf{Z}$-module. This clearly implies that $\xi_m(\tau)$ is an algebraic integer. Thus to prove that $\xi_m(\tau)$ is an integer, it suffices to see that $\xi_m(\tau)$ is a rational number. Let $d_{\tau,s}$ be the dimension of the space of s-invariants in τ. Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2 m_s (d_\tau - d_{\tau,s}) ,$$

which gives the rationality of $\xi_m(\tau)$. □