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$W_z$ -invariant, we deduce that  $p(x) - p(sx) = 0$ , so that in this case  $p(x) - p(sx)$  also is divisible by  $\alpha_s(x)^{2m_s+1}$ .

To conclude, notice that  $p(z) \neq 0$ . Indeed, for a reflection  $s$ ,  $\alpha_s$  vanishes exactly on the fixed points of  $s$ , so that  $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$ . Also for all  $w \in W_z$   $f(wz) = f(z) \neq 0$ . On the other hand, it is clear that  $p(y) = 0$ .  $\square$

EXAMPLE 1.5. Take  $W = \mathbf{Z}/2$ . As we have already seen,  $Q_m$  has a basis given by the monomials  $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$ . From this we deduce that setting  $z = x^2$  and  $y = x^{2m+1}$ ,  $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$ , where  $K$  is the plane curve with a cusp at the origin, given by the equation  $y^2 = z^{2m+1}$ . The map  $\pi: \mathbf{C} \rightarrow K$  is given by  $\pi(t) = (t^{2m+1}, t^2)$ , which is clearly bijective.

#### 1.4 FURTHER PROPERTIES OF $X_m$

Let us get to some deeper properties of quasi-invariants. Let  $X$  be an irreducible affine variety over  $\mathbf{C}$  and  $A = \mathbf{C}[X]$ . Recall that, by the Noether Normalization Lemma, there exist  $f_1, \dots, f_n \in \mathbf{C}[X]$  which are algebraically independent over  $\mathbf{C}$  and such that  $\mathbf{C}[X]$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ . This means that we have a finite morphism of  $X$  onto an affine space.

DEFINITION 1.6.  $A$  (and  $X$ ) is said to be *Cohen-Macaulay* if there exist  $f_1, \dots, f_n$  as above, with the property that  $\mathbf{C}[X]$  is a locally free module over  $\mathbf{C}[f_1, \dots, f_n]$ . (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that  $A$  is a free module.)

REMARK. If  $A$  is Cohen-Macaulay, then for any  $f_1, \dots, f_n$  which are algebraically independent over  $\mathbf{C}$  and such that  $A$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ , we have that  $A$  is a locally free  $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]).  $Q_m$  is Cohen-Macaulay.

Notice that, using Chevalley's result that  $\mathbf{C}[\mathfrak{h}]^W$  is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]).  $Q_m$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module.

We show how one can prove this Theorem in 3.10. This proof follows [BEG] (the original proof of [EG2] is shorter but somewhat less conceptual). The main idea of the proof is to show that the  $\mathbf{C}[\hbar]^W$ -module  $Q_m$  can be extended to a module over a bigger (noncommutative) algebra, namely the spherical subalgebra of the rational Cherednik algebra. Furthermore, this module belongs to an appropriate category of representations of this algebra, called category  $\mathcal{O}$ . On the other hand, it can be shown that any module over the spherical subalgebra that belongs to this category is free when restricted to the commutative algebra  $\mathbf{C}[\hbar]^W$ .

### 1.5 THE POINCARÉ SERIES OF $Q_m$

Consider now the Poincaré series

$$h_{Q_m}(t) = \sum_{r \geq 0} \dim Q_m[r] t^r,$$

where  $Q_m[r]$  denotes the graded component of  $Q_m$  of degree  $r$ . For every irreducible representation  $\tau \in \widehat{W}$ , define

$$\chi_\tau(t) = \sum_{r \geq 0} \dim \operatorname{Hom}_W(\tau, \mathbf{C}[\hbar][r]) t^r.$$

Consider the element in the group ring  $\mathbf{Z}[W]$

$$\mu_m = \sum_{s \in \Sigma} m_s (1 - s).$$

The  $W$ -invariance of  $m$  implies that  $\mu_m$  lies in the center of  $\mathbf{Z}[W]$ . Hence it is clear that  $\mu_m$  acts as a scalar,  $\xi_m(\tau)$ , on  $\tau$ . Let  $d_\tau$  be the degree of  $\tau$ .

LEMMA 1.9. *The scalar  $\xi_m(\tau)$  is an integer.*

*Proof.*  $\mathbf{Z}[W]$  and hence also its center, is a finite  $\mathbf{Z}$ -module. This clearly implies that  $\xi_m(\tau)$  is an algebraic integer. Thus to prove that  $\xi_m(\tau)$  is an integer, it suffices to see that  $\xi_m(\tau)$  is a rational number. Let  $d_{\tau,s}$  be the dimension of the space of  $s$ -invariants in  $\tau$ . Taking traces we get

$$d_\tau \xi_m(\tau) = \sum_{s \in \Sigma} 2m_s (d_\tau - d_{\tau,s}),$$

which gives the rationality of  $\xi_m(\tau)$ .  $\square$