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**Kapitel:** 2.2 The classical Calogero-Moser System

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called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x,p)$ on $T^*X$. Given $E$ and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x,p)}{dt} = \{f,E\}$. If $X$ is locally identified with $\mathbf{R}^n$ by choosing coordinates $x_1,\ldots,x_n$, then $T^*X$ is locally identified with $\mathbf{R}^{2n}$ with coordinates $x_1,\ldots,x_n,p_1,\ldots,p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x,p)$ is called an integral of motion for our system if $\{I,E\} = 0$. Integrals of motion are useful, since for any such integral $I$ the function $I(x(t),p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given $n$ functionally independent integrals of motion $I_1,\ldots,I_n$ with $\{I_l,I_k\} = 0$ for all $1 \leq l,k \leq n$, then all $2n$ variables $x_i,p_i$ can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

### 2.2 The classical Calogero-Moser system

Quasi-invariants are related to many-particle systems. Consider a system of $n$ particles on the real line $\mathbf{R}$. A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points $a$, $b$ have energy of interaction $U(a-b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i<j} U(x_i - x_j).$$

Here, $x_i$ are the coordinates of the particles, $p_i$ their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function $E$.

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

$\gamma$ being a constant.

The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group $W$ generated by reflections acting on the space $\mathfrak{h}$, and keep the notation of the previous section. Fix a $W$-invariant nondegenerate scalar product $(-,-)$ on $\mathfrak{h}$. It determines a scalar product on $\mathfrak{h}^*$. Define the "energy function"

$$E(x,p) = \frac{(p,p)}{2} + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$, where $\gamma\colon \Sigma \to \mathbf{C}$ is a $W$-invariant function. Notice that although $\alpha_s$ is defined up to a non zero constant, by homogeneity, $E$ is independent of the choice of $\alpha_s$. We will call the system defined by $E$ the Calogero-Moser system for $W$.

If $W$ is the symmetric group $S_n$, $\mathfrak{h} = \mathbf{C}^n$, then $\Sigma$ is the set of transpositions $s_{i,j}$, $i<j$, and we can take $\alpha_s = e_i - e_j$, Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for $W$ is completely integrable.

### 2.3 The quantum Calogero-Moser system

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy $E$ by formally making the substitution

$$p_j \Rightarrow -i\hbar\frac{\partial}{\partial x_j},$$

where $\hbar$ is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum_{s\in\Sigma} \frac{\gamma_s(\alpha_s,\alpha_s)}{\alpha_s^2},$$

where $\Delta$ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2}\Delta + \sum_{i<j} \frac{c}{(x_i - x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2}\widehat{E} = \Delta - \sum_{s\in\Sigma} \frac{\beta_s(\alpha_s,\alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.