Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 2.7 An example
DOI: https://doi.org/10.5169/seals-66677

The required extension to the ring of m-quasi-invariants is then provided by the following

THEOREM 2.10 ([CV1, CV2]). *Let* $c = m\colon \Sigma \to \mathbf{Z}_+$. *The following two conditions are equivalent for a homogeneous polynomial* $q \in \mathbf{C}[\mathfrak{h}^*]$ *of degree* d.

1) *There exists a differential operator*

$$L_q = q(\partial_{y_1}, \dots, \partial_{y_n}) + l.o.t.$$

of homogeneity degree $-d$, *such that* $[L_q, L] = 0$.

2) q *is an* m*-quasi-invariant homogeneous of degree* d.

Using this, we can extend system (5) to the system

$$L_p\psi = p(k)\psi\,, \quad p \in Q_m\,, \quad k \in \operatorname{Spec} Q_m = X_m\,. \tag{6}$$

(Recall that, as a set, $X_m = \mathfrak{h}$.) Near a generic point $x_0 \in \mathfrak{h}$, system (6) has a one dimensional space of solutions, thus there exists a unique up to scaling solution $\psi(k,x)$, which can be expressed in elementary functions. This solution is called the *Baker-Akhiezer function*, and has the form

$$\psi(k,x) = P(k,x)\, e^{(k,x)}$$

with $P(k,x)$ a polynomial of the form $\delta(x)\delta(k) + l.o.t.$ and $e^{(k,x)}$ denotes the exponential function computed in the scalar product (k,x). Furthermore, it can be shown that $\psi(k,x) = \psi(x,k)$ (see [CV1, CV2, FV]).

These results motivate the following terminology. The variety X_m is called *the spectral variety* of the Calogero-Moser system for the multiplicity function m, and Q_m is called *the spectral ring* of this system.

2.7 AN EXAMPLE

EXAMPLE 2.11. Let $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, $m = 1$. As we have seen, Q_m has a basis given by the monomials $\{x^{2i}\} \cup \{x^{2i+3}\}$, $i \geq 0$. Let us set for such a monomial, $L_{x^r} = L_r$, and $\partial = \frac{d}{dx}$. Then we have

$$L_0 = 1, \quad L_2 = \partial^2 - \frac{2}{x}\partial, \quad L_3 = \partial^3 - \frac{3}{x}\partial^2 + \frac{3}{x^2}\partial\,.$$

As for the others, $L_{2j} = L_2^j$, $L_{2j+3} = L_2^j L_3$. (Note that L_1 is not defined). The system (6) in this case is

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\[2mm] \psi''' - \dfrac{3}{x}\psi'' + \dfrac{3}{x^2}\psi' = k^3\psi\,. \end{cases}$$

The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \dfrac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\dfrac{1}{x} + k^2x)\psi' = -k^3x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate ψ_m for a general m.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m+1)(x\partial - 2m-1)\cdots(x\partial - 1)\,e^{kx}$.

Proof. We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m+1) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on m starting with $m=0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. □

3. LECTURE 3

3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique W-invariant