

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LECTURES ON QUASI-INVARIANTS OF COXETER GROUPS AND THE CHEREDNIK ALGEBRA
Autor: Etingof, Pavel / Strickland, Elisabetta
Kapitel: 3.4 The Cherednik algebra
DOI: <https://doi.org/10.5169/seals-66677>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 02.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3.4 THE CHEREDNIK ALGEBRA

Let us now return to the algebra \mathcal{A} of operators on U generated by $\mathcal{D}(U)$ and W . This algebra contains the Dunkl operators

$$D_y := \partial_y + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s} (s - 1).$$

LEMMA 3.10. *The following relations hold:*

$$\begin{aligned} [x_i, x_j] &= [D_{x_i}, D_{x_j}] = 0, \quad \forall 1 \leq i, j \leq n \\ [D_{x_i}, x_j] &= \delta_{ij} + \sum_{s \in \Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \leq i, j \leq n \end{aligned}$$

$$wxw^{-1} = w(x), \quad wD_yw^{-1} = D_{w(y)}, \quad \forall w \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

Proof. The proof is an easy computation, except for the relations $[D_{x_i}, D_{x_j}] = 0$, which follow from Theorem 2.6. \square

This lemma motivates the following definition.

DEFINITION 3.11 (see e.g. [EG]). The *Cherednik algebra* H_c is an associative algebra with generators $x_i, y_i, i = 1, \dots, n$, and $w \in W$, with defining relations

$$\begin{aligned} [x_i, x_j] &= [y_i, y_j] = 0, \quad \forall 1 \leq i, j \leq n \\ [y_i, x_j] &= \delta_{ij} + \sum_{s \in \Sigma} c_s \frac{(x_i, \alpha_s)(x_j, \alpha_s)}{(\alpha_s, \alpha_s)} s, \quad \forall 1 \leq i, j \leq n \end{aligned}$$

$$wxw^{-1} = w(x), \quad wyw^{-1} = w(y), \quad w \cdot w' = ww', \quad \forall w, w' \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

This algebra was introduced by Cherednik as a rational limit of his double affine Hecke algebra defined in [Ch]. Notice that if $c = 0$ then $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes \mathbf{C}[W]$.

Lemma 3.10 implies that the algebra H_c is equipped with a homomorphism $\phi: H_c \rightarrow \mathcal{A}$, given by $w \rightarrow w, x_i \rightarrow x_i, y_i \rightarrow D_{x_i}$.

Cherednik proved the following theorem.

THEOREM 3.12 (Poincaré-Birkhoff-Witt theorem). *The multiplication map*

$$\mu: \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[W] \rightarrow H_c$$

given by $\mu(f(x) \otimes g(y) \otimes w) = f(x)g(y)w$ is an isomorphism of vector spaces.

Proof. It is easy to see that the map μ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} w$ are linearly independent in H_c . To do this, it suffices to show that the images of these monomials under the homomorphism ϕ , i.e. $x_1^{i_1} \dots x_n^{i_n} D_{x_1}^{j_1} \dots D_{x_n}^{j_n} w$, are linearly independent.

Given an element $A \in \mathcal{A}$, writing $A = \sum_{w \in W} P_w w$ with $P_w \in \mathcal{D}(U)$ we define the order of A , $\text{ord}A$, as the maximum of the orders of the P_w 's. Notice that $\text{ord}AB \leq \text{ord}A + \text{ord}B$. We now remark that for any sequence of non negative indices (i_1, \dots, i_n) ,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

Indeed this is true for D_{x_i} . We proceed by induction on $r = i_1 + \dots + i_n$. We can clearly assume $i_1 > 0$, so by induction,

$$D_{x_1}^{i_1} \dots D_{x_n}^{i_n} = (\partial_{x_1} + l.o.t.)(\partial_{x_1}^{i_1-1} \dots \partial_{x_n}^{i_n} + l.o.t.) = \partial_{x_1}^{i_1} \dots \partial_{x_n}^{i_n} + l.o.t.$$

From this we deduce that for any pair of multiindices $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$, $w \in W$, setting $x_I = x_1^{i_1} \dots x_n^{i_n}$, $D_J = D_{x_1}^{j_1} \dots D_{x_n}^{j_n}$, $\partial_J = \partial_{x_1}^{j_1} \dots \partial_{x_n}^{j_n}$, we have

$$x_I D_J w = x_I \partial_J w + l.o.t.$$

Using this and the linear independence of the elements $x_I \partial_J w$, it is immediate to conclude that the elements $x_I D_J w$ are linearly independent, proving our claim. \square

REMARK 1. We see that the homomorphism ϕ identifies H_c with the subalgebra of \mathcal{A} generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators D_y , $y \in \mathfrak{h}$ and W .

REMARK 2. Another way to state the PBW theorem is the following. Let F^\bullet be a filtration on H_c defined by $\deg(x_i) = \deg(y_i) = 1$, $\deg(w) = 0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$ to the associated graded algebra $\text{gr}(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e = \frac{1}{W} \sum_{w \in W} w \in \mathbf{C}[W].$$