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SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS

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ABSTRACT. Let G_0 be a connected compact Lie group and let Γ be a finite group. Denote by $\mathcal{E}$ the set of equivalence classes of extensions of Γ by G_0. Using the notion of principal subgroup, we show that two nonconnected compact Lie groups are isomorphic if and only if the cohomology classes corresponding to their naturally associated extensions are in the same orbit under an action of $\mathrm{Out}(G_0) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Explicit examples of this cohomological classification are given. A revisited "Sandwich" Theorem and some criteria for the splitting of the extension associated to a compact Lie group are also presented.

1. INTRODUCTION

Naturally associated to a compact Lie group G, there is a group extension $G_0 \hookrightarrow G \twoheadrightarrow \Gamma$, where G_0 denotes the connected component of the identity of G, and $\Gamma = \pi_0(G) = G/G_0$ denotes the finite group of connected components. The problem we want to address here is the following: given a connected compact Lie group G_0 and a finite group Γ, can one classify *up to isomorphism* the compact Lie groups with connected component isomorphic to G_0 and with group of components isomorphic to Γ ? Of course, the classical theory of group extensions with nonabelian kernel and its relationship to the appropriate cohomology groups of degree 2 gives a partial answer to our question. However, these cohomology groups classify groups only *up to equivalence* of extensions, leaving the isomorphism question unsolved in general. This is illustrated, in the case of finite groups, by the following well-known example: the cohomology group $H^2(\mathbf{Z}/3;\mathbf{Z}/3)$ is isomorphic to $\mathbf{Z}/3$, but there are only two nonisomorphic groups with 9 elements, two

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