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Artikel: SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS
Autor: HÄMMERLI, Jean-François
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In the final section, we will see how the splitting of the extension associated to G is related to the splitting of the extensions associated to N and Q that appear in Theorem 2.2.

We end this section by recalling a very important result relating the inner, "usual", and outer automorphism groups of a connected compact Lie group. This result is one of the main reasons why the case of compact Lie groups is well controlled when applying the theory of group extensions, as we will see in Section 3. For the proof, we refer to de Siebenthal [9, Théorème, pp. 46–47] (for another approach consult Bourbaki [5], §4.10).

THEOREM 2.4 (de Siebenthal). *Let G_0 be a connected compact Lie group and let $H \subset G_0$ be a principal subgroup. Then the extension*

$$\mathrm{Inn}(G_0) \overset{\iota}{\hookrightarrow} \mathrm{Aut}(G_0) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_0)$$

is split, i.e.

$$\mathrm{Aut}(G_0) \cong \mathrm{Inn}(G_0) \rtimes \mathrm{Out}(G_0)\,.$$

A possible splitting is given by $s \colon \mathrm{Out}(G_0) \to \mathrm{Aut}(G_0)$, where $s(\alpha)$ is the unique automorphism in $\pi^{-1}(\alpha)$ fixing H pointwise.

REMARK 2.5. The fact that the extension associated to $\mathrm{Aut}(G_0)$ is split was known before the work of de Siebenthal, at least in the semisimple case, and appeared in a paper of Dynkin [10].

3. COMPACT LIE GROUPS AND EXTENSIONS

We assume knowledge of the classical relationship between group extensions and related cohomology groups of low degree, as first introduced by Eilenberg and Mac Lane [11]. For readers not familiar with it, the textbooks by Mac Lane [16], Robinson [21], or Adem-Milgram [2], provide a thorough treatment; a more concise approach can be found in Kirillov's book [15], and a sketch in Brown's [6]. We now want to apply this relationship to the case of compact Lie groups. We fix a nonabelian connected compact Lie group G_0, a finite group Γ, and a homomorphism $\varphi \colon \Gamma \to \mathrm{Out}(G_0)$. Recall that Z_0 denotes the center of G_0. Choosing a principal subgroup $H \subset G_0$ and fixing s as in Theorem 2.4, we get the commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\varphi} & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o) \\ & \searrow^{\bar{\varphi}} & \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ & & \mathrm{Aut}(Z_o) & = & \mathrm{Aut}(Z_o) \end{array}$$

In the sequel, we will use the notation $\sigma_\gamma = (s \circ \varphi)(\gamma)$, for $\gamma \in \Gamma$. Let $\mathcal{E}(\Gamma, G_o, \varphi) \subset \mathcal{E}$ denote the subset of equivalence classes giving rise to φ. In the particular case of compact Lie groups, one has the following results.

PROPOSITION 3.1.

(i) *The set of equivalence classes of extensions $\mathcal{E}(\Gamma, G_o, \varphi)$ is in bijection with the cohomology group $H^2_{\bar{\varphi}}(\Gamma; Z_o)$.*

(ii) *For all $u \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$ the corresponding extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ carries a natural structure of Lie group.*

Proof. It suffices to check that $\mathcal{E}(\Gamma, G_o, \varphi) \neq \varnothing$ to prove (i). But this follows from Theorem 2.4: the semidirect product $G = G_o \rtimes_{s\circ\varphi} \Gamma$ exists for any φ. The second statement is easily deduced from classical Lie group theory. □

The bijection in the latter proposition is not canonical, as it depends on the choice of a particular element in $\mathcal{E}(\Gamma, G_o, \varphi)$. On the other hand, there is a canonical bijection between $H^2_{\bar{\varphi}}(\Gamma; Z_o)$ and the set $\mathcal{E}(\Gamma, Z_o, \bar{\varphi})$ of equivalence classes of extension of Γ by Z_o with action given by $\bar{\varphi}$. Therefore, there is a bijection $\Lambda\colon \mathcal{E}(\Gamma, Z_o, \bar{\varphi}) \to \mathcal{E}(\Gamma, G_o, \varphi)$ still depending on the previous choice. Let us describe this bijection by first expliciting the cohomology group $H^2_{\bar{\varphi}}(\Gamma; Z_o) = Z^2_{\bar{\varphi}}(\Gamma; Z_o)/B^2_{\bar{\varphi}}(\Gamma; Z_o)$. Keeping the multiplicative notation in Z_o, the cocycles, i.e. the elements of $Z^2_{\bar{\varphi}}(\Gamma; Z_o)$, are functions $h\colon \Gamma \times \Gamma \to Z_o$ satisfying $h(\gamma_1, e) = h(e, \gamma_2) = e$ (normalization), and

$$(\delta h)(\gamma_1, \gamma_2, \gamma_3) = \sigma_{\gamma_1}(h(\gamma_2, \gamma_3)) \cdot h(\gamma_1\gamma_2, \gamma_3)^{-1} \cdot h(\gamma_1, \gamma_2\gamma_3) \cdot h(\gamma_1, \gamma_2)^{-1} = e$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. The coboundaries, i.e. the elements of $B^2_{\bar{\varphi}}(\Gamma; Z_o)$, are functions $h\colon \Gamma \times \Gamma \to Z_o$ such that there exists a function $k\colon \Gamma \to Z_o$, with $k(e) = e$, satisfying

$$h(\gamma_1, \gamma_2) = (\delta k)(\gamma_1, \gamma_2) = \sigma_{\gamma_1}(k(\gamma_2)) \cdot k(\gamma_1\gamma_2)^{-1} \cdot k(\gamma_1)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Let us choose the semidirect product $G_o \rtimes \Gamma$ associated to the section s as the extension corresponding to the neutral element in $H^2_{\bar{\varphi}}(\Gamma; Z_o)$. Then, for $u = [h] \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$, the corresponding class of extensions is given by $[G_o \hookrightarrow G_h \twoheadrightarrow \Gamma]$, where G_h is the *set* $G_o \times \Gamma$ equipped with the multiplication

$$(g,\gamma) *_h (g',\gamma') = \big(g \cdot \sigma_\gamma(g') \cdot h(\gamma,\gamma'), \gamma \cdot \gamma'\big)$$

(see [16], Chapter IV, §4 and §8). We will also denote by $G_o \hookrightarrow G_u \twoheadrightarrow \Gamma$ any representative of the class of extensions corresponding to $u \in H^2_{\bar\varphi}(\Gamma; Z_o)$. We now give a canonical description of the inverse of Λ, i.e. a description that does *not* depend on the choice of a particular element in $\mathcal{E}(\Gamma, G_o, \varphi)$.

LEMMA 3.2. *For any principal subgroup H in G_o, the map*

$$\Theta\colon \mathcal{E}(\Gamma, G_o, \varphi) \longrightarrow \mathcal{E}(\Gamma, Z_o, \bar\varphi),\ [G_o \hookrightarrow G \twoheadrightarrow \Gamma] \longmapsto [Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma]$$

is the inverse of Λ (and does not depend on the choice of H). In particular it is a bijection.

Proof. As centralizers of principal subgroups are preserved by isomorphisms of G, Θ does not depend on the choice of a representative in $[G_o \hookrightarrow G \twoheadrightarrow \Gamma]$. Let $u = [Z_o \hookrightarrow E_h \twoheadrightarrow \Gamma] = [h] \in H^2_{\bar\varphi}(\Gamma, Z_o)$. Then, we have the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E_h & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_o & \hookrightarrow & G = G_h & \twoheadrightarrow & \Gamma \end{array}$$

where E_h is $Z_o \times \Gamma$ as a set. Let us show that $E_h = Z_{G_h}(H)$, H being the principal subgroup of G_o corresponding to the fixed section s. By Theorem 2.2, it is enough to check that E_h is contained in $Z_{G_h}(H)$. Let $(z,\gamma) \in E_h$ and $(x,e) \in H \subset G_o \subset G_h$. We calculate

$$\begin{aligned}(z,\gamma) *_h (x,e) &= \big(z \cdot \sigma_\gamma(x) \cdot h(\gamma,e), \gamma\big)\\ &= (z \cdot x, \gamma),\end{aligned}$$

and

$$\begin{aligned}(x,e) *_h (z,\gamma) &= \big(x \cdot \sigma_e(z) \cdot h(e,\gamma), \gamma\big)\\ &= (x \cdot z, \gamma)\\ &= (z \cdot x, \gamma),\end{aligned}$$

by normalization, and because the restriction of σ_γ to H is the identity by the choice of the section s.

Now, as the principal subgroups are all conjugate by an element of G_o (see [8], Théorème, pp. 46–47), so are their centralizers. Therefore, the extensions $Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma$, for H running through the family of principal subgroups, all belong to the same class. This shows that Θ is well defined and satisfies $\Theta \circ \Lambda = id_{\mathcal{E}(\Gamma, Z_o, \bar{\varphi})}$. As Λ is bijective, this shows that $\Theta = \Lambda^{-1}$. □

We summarize the situation exposed in this section.

THEOREM 3.3. *Suppose given G_o, a homomorphism $\varphi \colon \Gamma \to \mathrm{Out}(G_o)$ and an extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$, for which the homomorphism $\Gamma \to \mathrm{Aut}(Z_o)$ coincides with $\bar{\varphi}$. Then, up to equivalence of extensions, there exists a unique compact Lie group G fitting into the commutative diagram*

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \downarrow \varphi \\
\mathrm{Inn}(G_o) & \hookrightarrow & \mathrm{Aut}(G_o) & \twoheadrightarrow & \mathrm{Out}(G_o)
\end{array}$$

where the rows are group extensions. Moreover the given data allow the construction of an extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, in which the subgroup Z is the centralizer of a principal subgroup.

Conversely, the class of the extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$ in $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ can be recovered by taking the centralizer of any principal subgroup.

4. PROOF OF THE MAIN THEOREM AND EXAMPLES

We are almost ready to show that the map described in the Introduction is an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on the set

$$\mathcal{E} \approx \coprod_{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Out}(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o)\,.$$

We first introduce some notation. For an element g in a group K, we will write c_g for conjugation by g, i.e. $c_g(x) = gxg^{-1}$, for all x in K. For $\alpha \in \mathrm{Out}(G_o)$, we will choose $\widetilde{\alpha} \in \mathrm{Aut}(G_o)$ such that $\pi(\widetilde{\alpha}) = \alpha$, and we will denote the restricted automorphism by $\bar{\alpha} \in \mathrm{Aut}(Z_o)$. Finally,