

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS
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Kapitel: 5. Splitting of the extension ASSOCIATED TO A NONCONNECTED COMPACT LIE GROUP
DOI: <https://doi.org/10.5169/seals-66678>

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5. SPLITTING OF THE EXTENSION
ASSOCIATED TO A NONCONNECTED COMPACT LIE GROUP

Let G still denote a compact Lie group with a nonabelian connected component; we also assume the other notations introduced previously. In this final section, we make a few observations on the following problem: when is the natural extension associated to G split, i.e. when is G isomorphic to a semidirect product $G_0 \rtimes \Gamma$? Our aim is to relate this problem to the rest of this work. For a deeper analysis one should consult Chapter 6 in the book by Hofmann and Morris [14]. We start with a structure theorem for compact Lie groups based on centralizers of principal subgroups, similar to the “Sandwich Theorem for compact Lie groups” (see [14], Corollary 6.75, p.272). This theorem shows that any compact Lie group is “sandwiched” in between two semidirect products closely related to it. We then recall a theorem of de Siebenthal and compare, in some particular cases, the question of the splitting of the extension associated to G to that of the extensions associated to the normalizer N of a maximal torus and to the centralizer Q of a principal diagonal, both introduced in Section 2. As an application of the “Sandwich” Theorem, the final proposition presents a “minimal” compact Lie group G such that the associated extension is *not* split.

Let \bar{G}_0 denote the adjoint group G_0/Z_0 ; it is well-known that the center of \bar{G}_0 is trivial.

THEOREM 5.1. *Let G be a compact Lie group with a nonabelian connected component. Then there exist two surjective homomorphisms*

$$\begin{array}{ccc} G_s = G_0 \rtimes Z & \xrightarrow{\pi_1} & G & \xrightarrow{\pi_2} & \bar{G} = G/Z_0 \cong \bar{G}_0 \rtimes \Gamma, \\ (g_0, z) & \longmapsto & g_0 \cdot z & & \end{array}$$

where the centralizer Z of a fixed principal subgroup acts on G_0 by conjugation, and where π_2 is the canonical projection corresponding to the normal subgroup Z_0 of G .

For the kernels, we have $\ker \pi_1 \cong Z_0$ and $\ker \pi_2 = Z_0$; in particular

$$G \cong (G_0 \rtimes Z)/Z_0 \quad \text{and} \quad G/Z_0 \cong \bar{G}_0 \rtimes \Gamma.$$

Proof. As \bar{G}_0 is centerless, \bar{G} must be isomorphic to $\bar{G}_0 \rtimes \Gamma$ by the proof of Proposition 3.1. The other assertions about π_2 are clear.

The map π_1 is well-defined and surjective (because Z intersects every component of G). Straightforward computations show that it is a homomorphism and that $\ker \pi_1 \cong Z_0$. \square

REMARK 5.2. 1) The component of the identity of G_s is equal to G_o if and only if G_o is semisimple.

2) The present version of the "Sandwich" Theorem has the advantage of being more explicit than the one in [14] (the result therein is an existence theorem). Its drawbacks are the fact that Z is not finite if G_o is not semisimple, and that it obviously makes no sense for compact Lie groups with an abelian connected component.

3) Given a homomorphism $\varphi: \Gamma \rightarrow \text{Out}(G_o)$, and an extension $Z_o \xrightarrow{\mu} E \xrightarrow{\nu} \Gamma$ for which the action coincides with the "restriction" $\bar{\varphi}$, there is a more direct way than the cohomological one to recover the corresponding compact Lie group G , i.e. the one that fits into the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \parallel \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

Let us define the composition $\bar{\sigma}: E \xrightarrow{\nu} \Gamma \xrightarrow{\varphi} \text{Out}(G_o) \xrightarrow{s} \text{Aut}(G_o)$, where s is as in Theorem 2.4. Then by Bourbaki (see [4], Lemme 7, pp. 210–211), we have

$$G = (G_o \rtimes_{\bar{\sigma}} E) / \Delta Z_o,$$

where ΔZ_o is the image of the injection $z_o \mapsto (z_o^{-1}, \mu(z_o))$. Taking $E = Z$, this gives another proof of the assertions concerning π_1 in Theorem 5.1.

Using Cartan subgroups (in the sense of Segal [22], i.e. those Adams called "SS subgroups" in honour of Segal and de Siebenthal [1]), de Siebenthal gave some explicit sufficient conditions for the splitting of the extension associated to G ([9], Théorème p. 74).

THEOREM 5.3 (de Siebenthal). *Let G be a compact Lie group with G_o simply connected, or of adjoint type (i.e. Z_o is trivial). If $\Gamma = \pi_o(G)$ is cyclic then G is a semidirect product, i.e. $G \cong G_o \rtimes \Gamma$.*

A relationship with the splitting of the extension associated to the normalizer of a maximal torus N in G is given in the next proposition.

PROPOSITION 5.4. *If the group of components Γ of G is nilpotent, then the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ is split.*

Proof. The “if” part is clear. Conversely, let $s: \Gamma \rightarrow G$ be a section. By a result in Bourbaki (see [5], Corollaire 4, p. 49), any nilpotent subgroup of a compact Lie group is contained in the normalizer of some maximal torus. Therefore, if needed after conjugation by an element in G_0 , we have $s(\Gamma) \subset N$, and we can conclude that the extension associated to N is split. \square

For an extended maximal torus Q , the extensions can be related as follows.

PROPOSITION 5.5. *If the group of components Γ of G is cyclic, then the extension $G_0 \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $T \hookrightarrow Q \twoheadrightarrow \Gamma$ is split.*

Proof. The proposition readily follows from the fact that the conjugates of Q cover G . \square

REMARK 5.6. This latter proposition fails in general. An obstruction to the splitting of the extension associated to the extended maximal torus Q can be found in a paper by Oliver; this obstruction involves the representation ring of G and its relation with the family of all p -toral subgroups of G (see [19], Corollary 3.11). In particular, Oliver constructs a compact Lie group $G = \text{SU}(2) \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/3)$ such that the extension corresponding to the extended maximal torus Q is *not* split ([19], pp. 376–377).

We conclude with the promised example.

PROPOSITION 5.7. *Let $D_8 = \langle r, s \mid r^4 = s^2 = e, srs^{-1} = r^{-1} \rangle$ be a presentation of the dihedral group. Then the quotient*

$$G = (\text{SU}(2) \times D_8) / \Delta\mathbf{Z}/2,$$

where $\Delta\mathbf{Z}/2$ denotes the central subgroup generated by $(-1, r^2)$, is a compact Lie group with $G_0 \cong \text{SU}(2)$ and $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$, the 4-group of Klein. The associated extension

$$\text{SU}(2) \hookrightarrow G \twoheadrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is not split. Among the extensions associated to a compact Lie group with a nonabelian connected component, it is a minimal one having the property of being non-split, in the sense that the rank of the connected component and the size of the group of components are minimal. Moreover replacing the

connected component by $SO(3)$ (i.e. by “any” group of the same rank), or the group of components by $\mathbf{Z}/4$ (i.e. by “any” group of the same size), will force the extension to be split.

Proof. The assertions about the connected component and the group of components are clear. Let us show that the extension associated to G is not split. Let us denote $[g, \gamma] \in G$ the image of $(g, \gamma) \in SU(2) \times D_8$ under the canonical projection. Let \mathbf{S}^1 denote the standard maximal torus in $SU(2)$, and let N denote its normalizer in G . We have

$$N = \{[t, e] : t \in \mathbf{S}^1\} \amalg \{[jt, e] : t \in \mathbf{S}^1\} \amalg \{[t, r] : t \in \mathbf{S}^1\} \amalg \{[jt, r] : t \in \mathbf{S}^1\} \\ \amalg \{[t, s] : t \in \mathbf{S}^1\} \amalg \{[jt, s] : t \in \mathbf{S}^1\} \amalg \{[t, rs] : t \in \mathbf{S}^1\} \amalg \{[jt, rs] : t \in \mathbf{S}^1\}.$$

By contradiction, suppose that the extension associated to G is split, i.e. there exists a section. As $\mathbf{Z}/2 \times \mathbf{Z}/2$ is abelian, thus nilpotent, we deduce, by Proposition 5.4, that the extension associated to N is also split. We want to show that this is not possible by considering the elements of order 2 in N . For $n = 0, 1$, a straightforward calculation shows that in the component corresponding to $r^n s$, an element $[t, r^n s]$ is of order 2 if and only if $t = \pm \mathbf{1}$, and that the sub-component $\{[jt, r^n s] : t \in \mathbf{S}^1\}$ does not contain any element of order 2. Two of the three non-trivial elements in $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ must thus be mapped by the section to $[\pm \mathbf{1}, s]$ and $[\pm \mathbf{1}, rs]$. Therefore, as the section is a homomorphism, the image of the third non-trivial element is

$$[\pm \mathbf{1}, rs] \cdot [\pm \mathbf{1}, s] = [\pm \mathbf{1}, r],$$

which is not of order 2. A contradiction that shows that the extension associated to G is not split.

The property of minimality follows by Theorem 5.3, and by the fact that any extension with $SO(3)$ as normal subgroup is a direct product (because $SO(3)$ is complete, i.e. centerless and with trivial outer automorphism group). \square

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