Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 49 (2003)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS
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Bibliographie
DOI: https://doi.org/10.5169/seals-66678

connected component by SO(3) *(i.e. by "any" group of the same rank), or the group of components by* $\mathbf{Z}/4$ *(i.e. by "any" group of the same size), will force the extension to be split.*

Proof. The assertions about the connected component and the group of components are clear. Let us show that the extension associated to G is not split. Let us denote $[g,\gamma] \in G$ the image of $(g,\gamma) \in \mathrm{SU}(2) \times \mathrm{D}_8$ under the canonical projection. Let $\mathbf{S}^1$ denote the standard maximal torus in SU(2), and let N denote its normalizer in G. We have

$$N = \left\{[t,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,r] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,r] : t \in \mathbf{S}^1\right\}$$
$$\amalg \left\{[t,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,rs] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,rs] : t \in \mathbf{S}^1\right\}.$$

By contradiction, suppose that the extension associated to G is split, i.e. there exists a section. As $\mathbf{Z}/2 \times \mathbf{Z}/2$ is abelian, thus nilpotent, we deduce, by Proposition 5.4, that the extension associated to N is also split. We want to show that this is not possible by considering the elements of order 2 in N. For $n = 0, 1$, a straightforward calculation shows that in the component corresponding to $r^n s$, an element $[t, r^n s]$ is of order 2 if and only if $t = \pm\mathbf{1}$, and that the sub-component $\left\{[jt, r^n s] : t \in \mathbf{S}^1\right\}$ does not contain any element of order 2. Two of the three non-trivial elements in $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ must thus be mapped by the section to $[\pm\mathbf{1}, s]$ and $[\pm\mathbf{1}, rs]$. Therefore, as the section is a homomorphism, the image of the third non-trivial element is

$$[\pm\mathbf{1}, rs] \cdot [\pm\mathbf{1}, s] = [\pm\mathbf{1}, r] ,$$

which is not of order 2. A contradiction that shows that the extension associated to G is not split.

The property of minimality follows by Theorem 5.3, and by the fact that any extension with SO(3) as normal subgroup is a direct product (because SO(3) is complete, i.e. centerless and with trivial outer automorphism group). □

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(Reçu le 25 juin 2002)

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