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ATIYAH'S L^2-INDEX THEOREM

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1. INTRODUCTION

The L^2-Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold M in terms of the G-equivariant index of some regular covering $\widetilde{M}$ of M, with G the group of covering transformations. Atiyah's proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

2. REVIEW OF THE L^2-INDEX THEOREM

The main reference for this section is Atiyah's paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let M be a closed manifold and let E, F denote two complex (Hermitian) vector bundles over M. Consider an elliptic pseudo-differential operator

$$D\colon C^\infty(M,E) \to C^\infty(M,F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{s \in C^\infty(M,E) \mid Ds = 0\}\ .$$

The complex vector space S_D has finite dimension (see [13]), and so has S_{D^*} the space of solutions of the adjoint D^* of D where

$$D^*\colon C^\infty(M,F) \to C^\infty(M,E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_F \, dm = \langle s, D^* s' \rangle = \int_M \langle s(m), D^* s'(m) \rangle_E \, dm$$

for all $s \in C^\infty(M, E)$, $s' \in C^\infty(M, F)$. One now defines the *index* of D as follows:

$$\mathrm{Index}(D) = \dim_{\mathbf{C}}(S_D) - \dim_{\mathbf{C}}(S_{D^*}) \in \mathbf{Z}.$$

An explicit formula for Index(D) is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering $\pi\colon \widetilde{M} \to M$ with countable covering transformation group G. The projection π can be used to define an elliptic operator

$$\widetilde{D} := \pi^*(D)\colon C_c^\infty(\widetilde{M}, \pi^* E) \to C_c^\infty(\widetilde{M}, \pi^* F).$$

Denote by $S_{\widetilde{D}}$ the closure of $\left\{ s \in C_c^\infty(\widetilde{M}, \pi^* E) \mid \widetilde{D} s = 0 \right\}$ in $L^2(\widetilde{M}, \pi^* E)$. Let $\widetilde{D}^*$ denote the adjoint of $\widetilde{D}$. The space $S_{\widetilde{D}}$ is not necessarily finite dimensional, but being a closed G-invariant subspace of the L^2-completion $L^2(\widetilde{M}, \pi^* E)$ of the space of smooth sections with compact supports $C_c^\infty(\widetilde{M}, \pi^* E)$, its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \left\{ P\colon \ell^2(G) \to \ell^2(G) \text{ bounded and } G\text{-invariant} \right\}$$

for the group von Neumann algebra of G, where G acts on $\ell^2(G)$ via the right regular representation. Then $S_{\widetilde{D}}$ is a finitely generated Hilbert G-module and hence can be represented by an idempotent matrix $P = (p_{ij}) \in M_n(\mathcal{N}(G))$ (recall that a finitely generated Hilbert G-module is isometrically G-isomorphic to a Hilbert G-subspace of the Hilbert space $\ell^2(G)^n$ for some $n \geq 1$, see [9]). One then sets

$$\dim_G(S_{\widetilde{D}}) = \sum_{i=1}^{n} \langle p_{ii}(e), e \rangle = \kappa(P) \in \mathbf{R},$$

where by abuse of notation e denotes the element in $\ell^2(G)$ taking value 1 on the neutral element $e \in G$ and 0 elsewhere (see Eckmann's survey [9] on L^2-cohomology for more on von Neumann dimensions). The map $\kappa\colon M_n(\mathcal{N}(G)) \to \mathbf{C}$ is the Kaplansky trace. One defines the L^2-*index* of $\widetilde{D}$ by

$$\mathrm{Index}_G(\widetilde{D}) = \dim_G(S_{\widetilde{D}}) - \dim_G(S_{\widetilde{D}^*}).$$

We can now state Atiyah's L^2-Index Theorem.

THEOREM 2.1 (Atiyah [1]). *For D an elliptic pseudo-differential operator on a closed Riemannian manifold M*

$$\mathrm{Index}(D) = \mathrm{Index}_G(\widetilde{D})$$

for any countable group G and any lift $\widetilde{D}$ of D to a regular G-cover $\widetilde{M}$ of M.

In particular, the L^2-index of $\widetilde{D}$ is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the L^2-Index Theorem.

EXAMPLE 2.2 (Atiyah's formula [1]). Let $\Omega^\bullet$ be the de Rham complex of complex valued differential forms on the closed connected manifold M and consider the de Rham differential $D = d + d^* : \Omega^{ev} \to \Omega^{odd}$. Let $\pi : \widetilde{M} \to M$ be the universal cover of M so that $G = \pi_1(M)$. Then

- $\mathrm{Index}(D) = \chi(M)$, the ordinary Euler characteristic of M.
- $\mathrm{Index}_G(\widetilde{D}) = \sum_j (-1)^j \beta^j(M)$, the L^2-Euler characteristic of M.

The $\beta^j(M)$'s denote the L^2-Betti numbers of M. Thus the L^2-Index Theorem translates into Atiyah's formula

$$\chi(M) = \sum_j (-1)^j \beta^j(M) .$$

We recall that the L^2-Betti numbers $\beta^j(M)$ are in general not integers. For instance, if $\pi_1(M)$ is a finite group, one checks that

$$\beta^j(M) = \frac{1}{|\pi_1(M)|} b^j(\widetilde{M}) ,$$

where $b^j(\widetilde{M})$ stands for the ordinary j'th Betti number of the universal cover $\widetilde{M}$ of M. In particular, for $1 < |\pi_1(M)| < \infty$, $\beta^0(M) = 1/|\pi_1(M)|$ is not an integer and the L^2-Index Theorem reduces to the well-known fact that

$$\chi(M) = \frac{\chi(\widetilde{M})}{|\pi_1(M)|} .$$

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold M the L^2-Betti numbers $\beta^j(M)$ are always rational numbers, and even integers in case that $\pi_1(M)$ is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].